Heat and Mass Transfer: Fundamentals & Applications
Fourth Edition
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Chapter 4

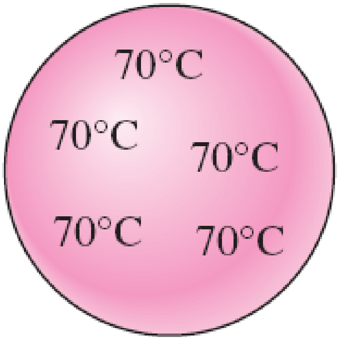
TRANSIENT HEAT CONDUCTION

LUMPED SYSTEM ANALYSIS

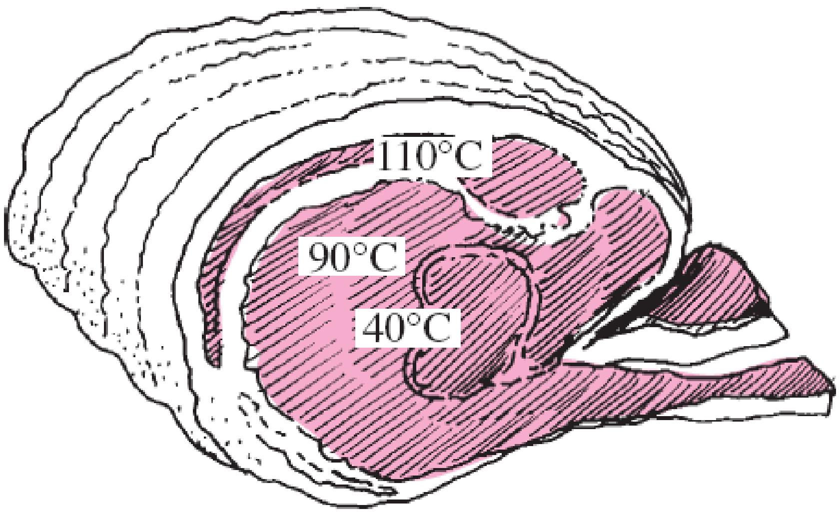
Interior temperature of some bodies remains essentially uniform at all times during a heat transfer process.

The temperature of such bodies can be taken to be a function of time only, *T*(*t*).

Heat transfer analysis that utilizes this idealization is known as **lumped system analysis**.



(*a*) Copper ball



(*b*) Roast beef

A small copper ball can be modeled as a lumped system, but a roast beef cannot.

$$\begin{pmatrix}\text{Heat transfer into the body} \\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The increase in the} \\ \text{energy of the body} \\ \text{during } dt\end{pmatrix}$$

$$hA_s(T_\infty - T)\,dt = mc_p\,dT$$

$$m = \rho V \qquad dT = d(T - T_\infty)$$

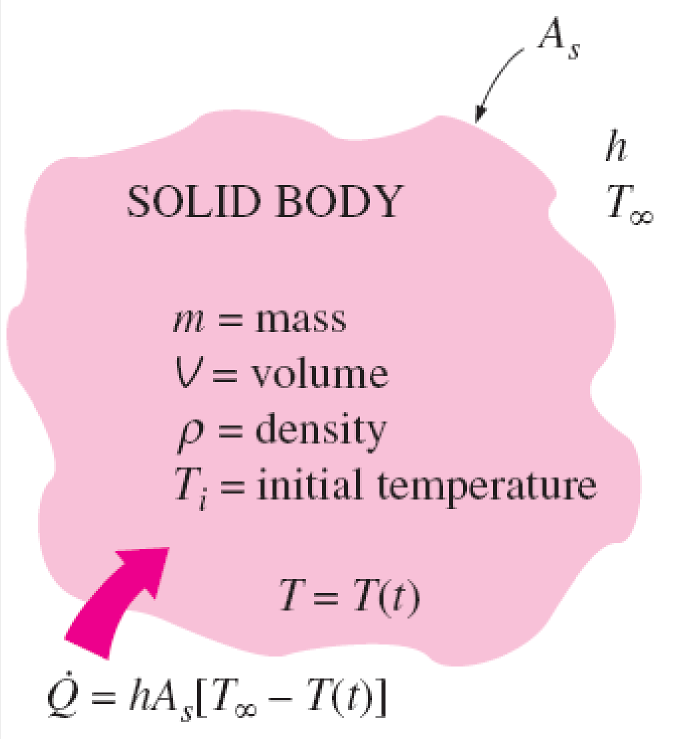
$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho V c_p}\,dt$$

Integrating with
$T = T_i$ at $t = 0$
$T = T(t)$ at $t = t$

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho V c_p}\,t$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad b = \frac{hA_s}{\rho V c_p} \quad (1/\mathrm{s})$$

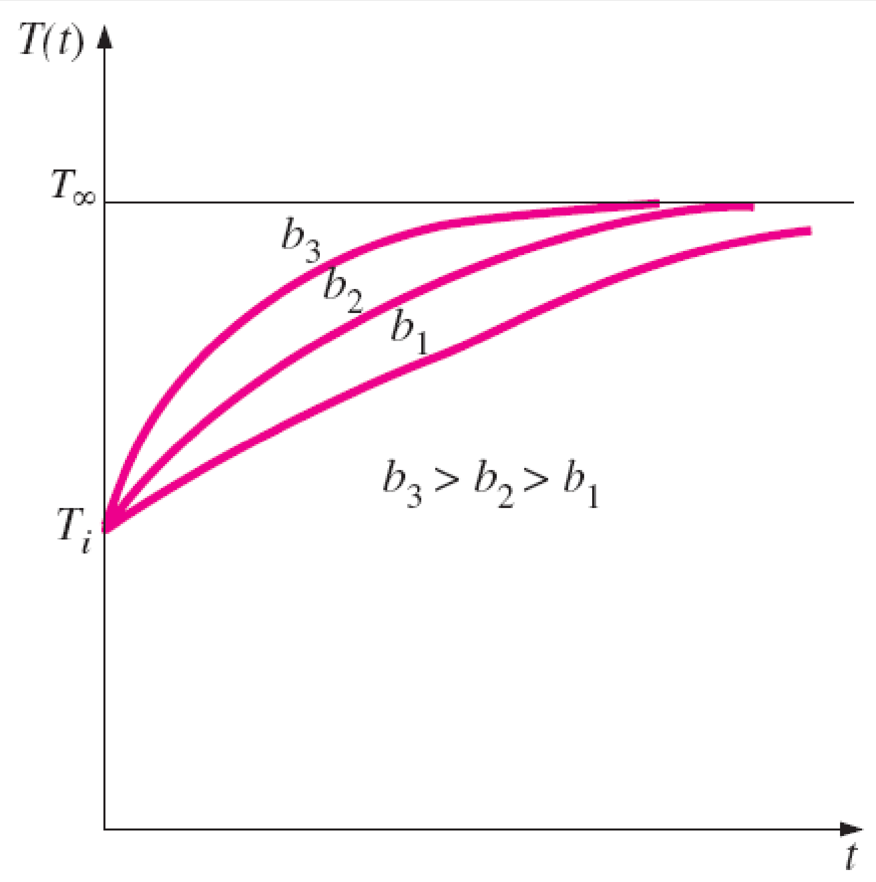
time constant



The geometry and parameters involved in the lumped system analysis.

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA_s}{\rho V c_p}$$



The temperature of a lumped system approaches the environment temperature as time gets larger.

- This equation enables us to determine the temperature *T*(*t*) of a body at time *t*, or alternatively, the time *t* required for the temperature to reach a specified value *T*(*t*).
- The temperature of a body approaches the ambient temperature T_∞ exponentially.
- The temperature of the body changes rapidly at the beginning, but rather slowly later on. A large value of *b* indicates that the body approaches the environment temperature in a short time

$$\dot{Q}(t) = hA_s[T(t) - T_\infty] \quad \text{(W)}$$

The *rate* of convection heat transfer between the body and its environment at time *t*

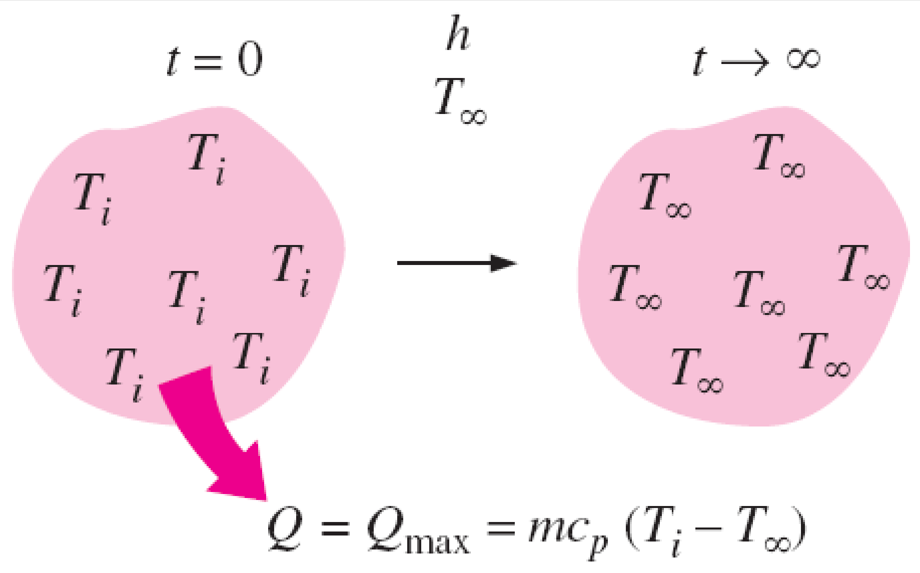
$$Q = mc_p[T(t) - T_i] \quad \text{(kJ)}$$

The *total amount* of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to t

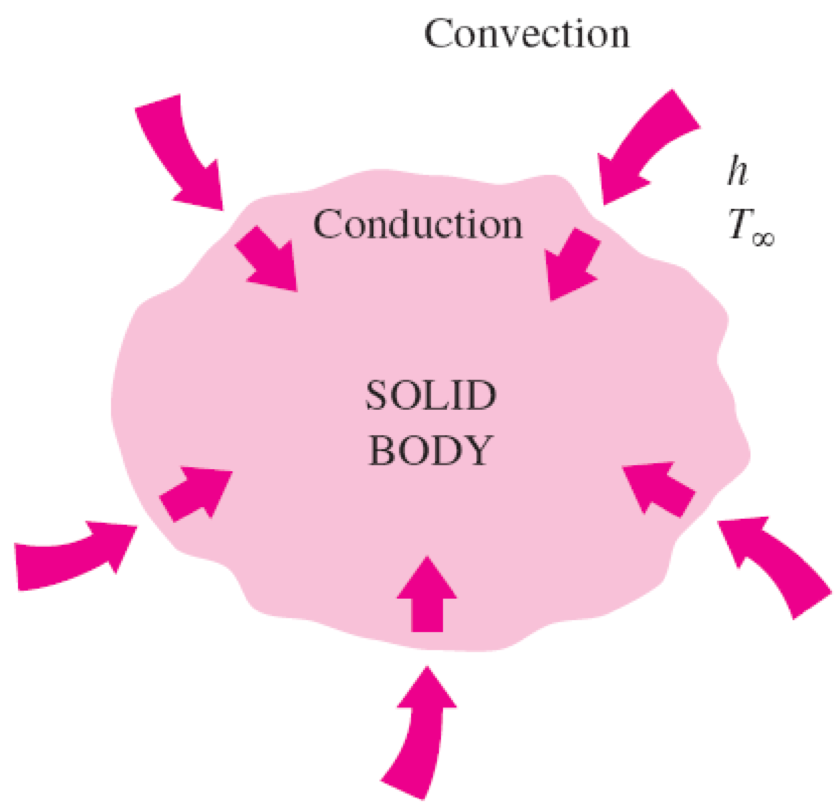
$$Q_{\max} = mc_p(T_\infty - T_i) \quad \text{(kJ)}$$

The *maximum* heat transfer between the body and its surroundings

Heat transfer to or from a body reaches its maximum value when the body reaches the environment temperature.



Criteria for Lumped System Analysis



$$\mathrm{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$

$$L_c = \frac{V}{A_s}$$ Characteristic length

$$\mathrm{Bi} = \frac{hL_c}{k}$$ Biot number

Lumped system analysis is *applicable* if

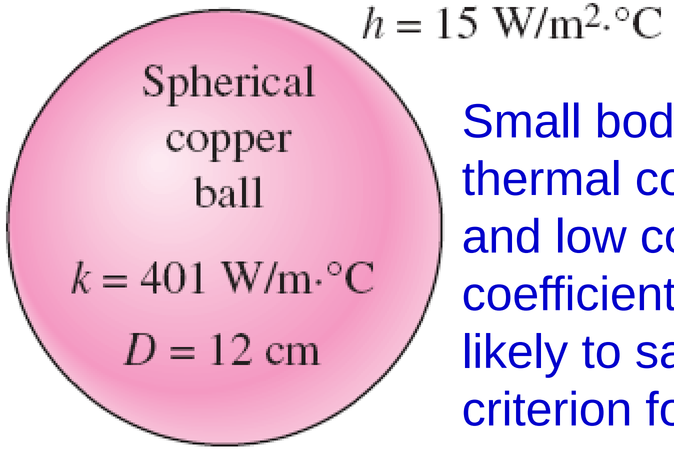
$$\mathrm{Bi} \leq 0.1$$

When Bi ≤ 0.1, the temperatures within the body relative to the surroundings (i.e., $T - T_\infty$) remain within 5 percent of each other.

$$\mathrm{Bi} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

$$\mathrm{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$

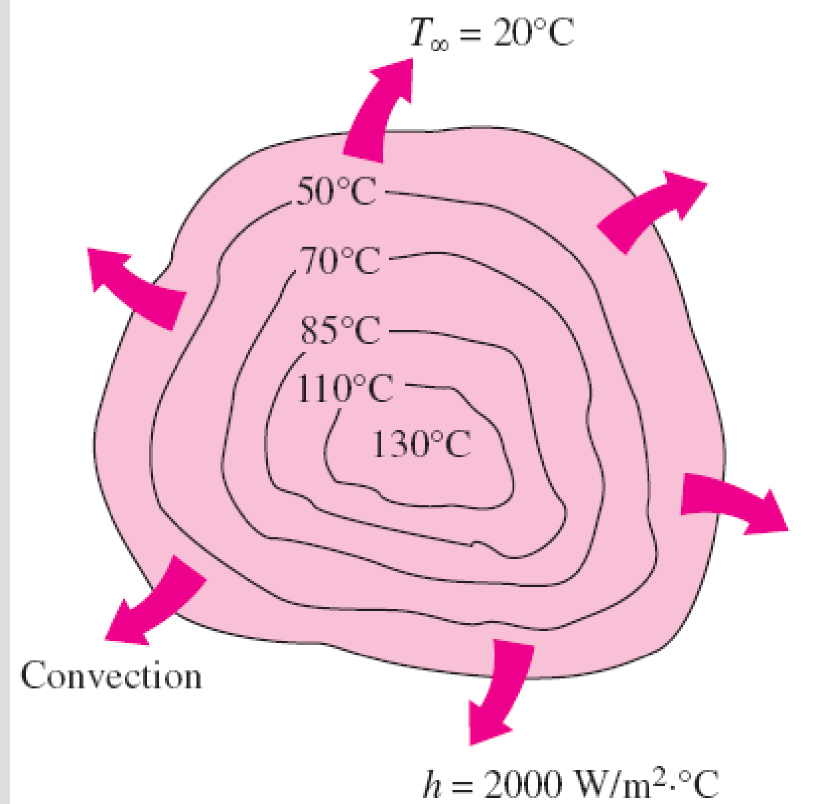
Jean-Baptiste Biot (1774–1862) was a French physicist, astronomer, and mathematician born in Paris, France. Although younger, Biot worked on the analysis of heat conduction even earlier than Fourier did (1802 or 1803) and attempted, unsuccessfully, to deal with the problem of incorporating external convection effects in heat conduction analysis. Fourier read Biot's work and by 1807 had determined for himself how to solve the elusive problem. In 1804, Biot accompanied Gay Lussac on the first balloon ascent undertaken for scientific purposes. In 1820, with Felix Savart, he discovered the law known as "Biot and Savart's Law." He was especially interested in questions relating to the polarization of light, and for his achievements in this field he was awarded the Rumford Medal of the Royal Society in 1840. The dimensionless **Biot number (Bi)** used in transient heat transfer calculations is named after him.



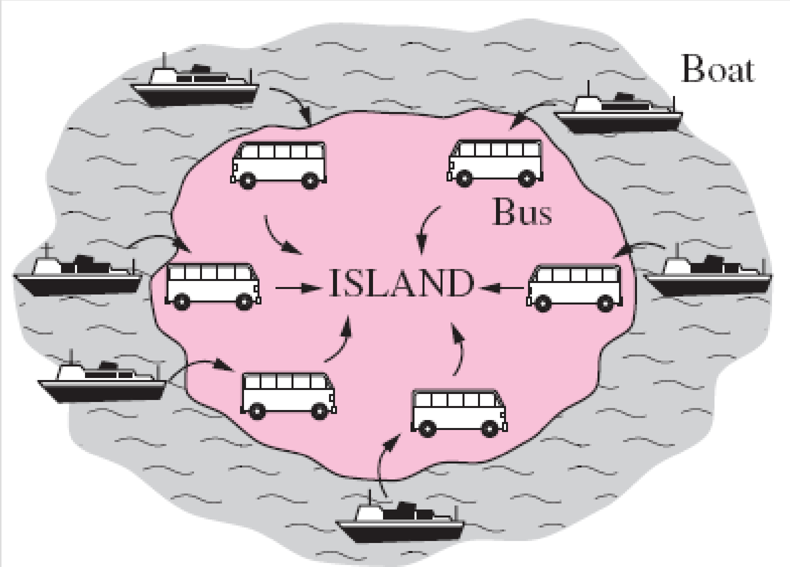
Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.

$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02 \text{ m}$$

$$\text{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$



When the convection coefficient *h* is high and *k* is low, large temperature differences occur between the inner and outer regions of a large solid.



Analogy between heat transfer to a solid and passenger traffic to an island.

EXAMPLE 4–1 **Temperature Measurement by Thermocouples**

The temperature of a gas stream is to be measured by a thermocouple whose junction can be approximated as a 1-mm-diameter sphere, as shown in Fig. 4–9. The properties of the junction are $k = 35$ W/m · °C, $\rho = 8500$ kg/m^3, and $C_p = 320$ J/kg · °C, and the convection heat transfer coefficient between the junction and the gas is $h = 210$ W/m^2 · °C. Determine how long it will take for the thermocouple to read 99 percent of the initial temperature difference.

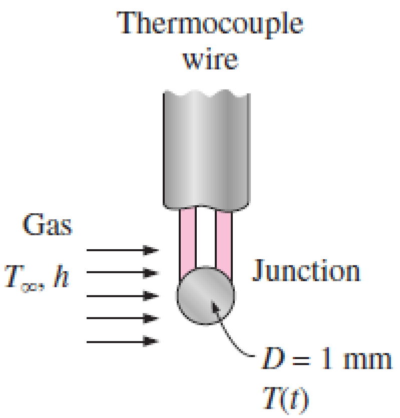


FIGURE 4–9
Schematic for Example 4–1.

SOLUTION The temperature of a gas stream is to be measured by a thermocouple. The time it takes to register 99 percent of the initial ΔT is to be determined.

Assumptions **1** The junction is spherical in shape with a diameter of $D = 0.001$ m. **2** The thermal properties of the junction and the heat transfer coefficient are constant. **3** Radiation effects are negligible.

Properties The properties of the junction are given in the problem statement.

Analysis The characteristic length of the junction is

$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = \frac{1}{6}(0.001 \text{ m}) = 1.67 \times 10^{-4} \text{ m}$$

Then the Biot number becomes

$$\text{Bi} = \frac{hL_c}{k} = \frac{(210 \text{ W/m}^2 \cdot {}^\circ\text{C})(1.67 \times 10^{-4} \text{ m})}{35 \text{ W/m} \cdot {}^\circ\text{C}} = 0.001 < 0.1$$

Therefore, lumped system analysis is applicable, and the error involved in this approximation is negligible.

In order to read 99 percent of the initial temperature difference $T_i - T_\infty$ between the junction and the gas, we must have

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = 0.01$$

For example, when $T_i = 0°\text{C}$ and $T_\infty = 100°\text{C}$, a thermocouple is considered to have read 99 percent of this applied temperature difference when its reading indicates $T(t) = 99°\text{C}$.

The value of the exponent b is

$$b = \frac{hA_s}{\rho C_p V} = \frac{h}{\rho C_p L_c} = \frac{210 \text{ W/m}^2 \cdot {}^\circ\text{C}}{(8500 \text{ kg/m}^3)(320 \text{ J/kg} \cdot {}^\circ\text{C})(1.67 \times 10^{-4} \text{ m})} = 0.462 \text{ s}^{-1}$$

We now substitute these values into Eq. 4–4 and obtain

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow 0.01 = e^{-(0.462 \text{ s}^{-1})t}$$

which yields

$$t = 10 \text{ s}$$

Therefore, we must wait at least 10 s for the temperature of the thermocouple junction to approach within 1 percent of the initial junction-gas temperature difference.

Discussion Note that conduction through the wires and radiation exchange with the surrounding surfaces will affect the result, and should be considered in a more refined analysis.

EXAMPLE 4–2 **Predicting the Time of Death**

A person is found dead at 5 PM in a room whose temperature is 20°C. The temperature of the body is measured to be 25°C when found, and the heat transfer coefficient is estimated to be $h = 8$ W/m$^2 \cdot$ °C. Modeling the body as a 30-cm-diameter, 1.70-m-long cylinder, estimate the time of death of that person (Fig. 4–10).

FIGURE 4–10
Schematic for Example 4–2.

SOLUTION A body is found while still warm. The time of death is to be estimated.

Assumptions **1** The body can be modeled as a 30-cm-diameter, 1.70-m-long cylinder. **2** The thermal properties of the body and the heat transfer coefficient are constant. **3** The radiation effects are negligible. **4** The person was healthy(!) when he or she died with a body temperature of 37°C.

Properties The average human body is 72 percent water by mass, and thus we can assume the body to have the properties of water at the average temperature of (37 + 25)/2 = 31°C; k = 0.617 W/m · °C, ρ = 996 kg/m^3, and C_p = 4178 J/kg · °C (Table A-9).

Analysis The characteristic length of the body is

$$L_c = \frac{V}{A_s} = \frac{\pi r_o^2 L}{2\pi r_o L + 2\pi r_o^2} = \frac{\pi(0.15 \text{ m})^2(1.7 \text{ m})}{2\pi(0.15 \text{ m})(1.7 \text{ m}) + 2\pi(0.15 \text{ m})^2} = 0.0689 \text{ m}$$

Then the Biot number becomes

$$\text{Bi} = \frac{hL_c}{k} = \frac{(8 \text{ W/m}^2 \cdot {}^\circ\text{C})(0.0689 \text{ m})}{0.617 \text{ W/m} \cdot {}^\circ\text{C}} = 0.89 > 0.1$$

Therefore, lumped system analysis is *not* applicable. However, we can still use it to get a "rough" estimate of the time of death. The exponent b in this case is

$$b = \frac{hA_s}{\rho C_p V} = \frac{h}{\rho C_p L_c} = \frac{8 \text{ W/m}^2 \cdot {}^\circ\text{C}}{(996 \text{ kg/m}^3)(4178 \text{ J/kg} \cdot {}^\circ\text{C})(0.0689 \text{ m})}$$
$$= 2.79 \times 10^{-5} \text{ s}^{-1}$$

We now substitute these values into Eq. 4–4,

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow \frac{25 - 20}{37 - 20} = e^{-(2.79 \times 10^{-5} \text{ s}^{-1})t}$$

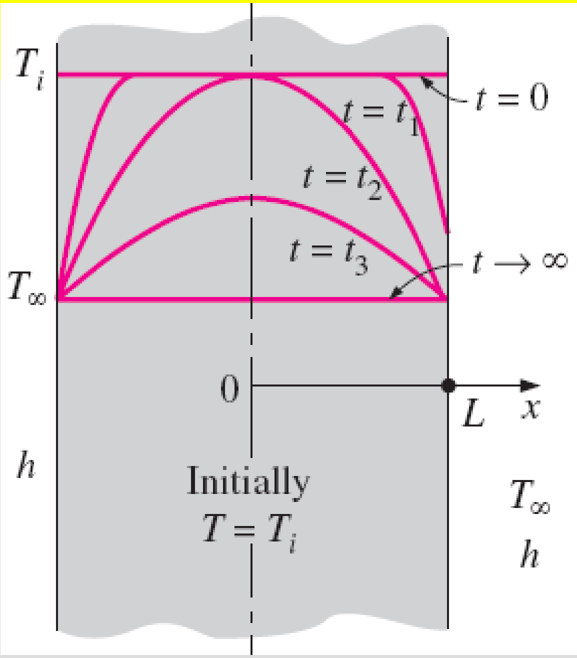
which yields

$$t = 43{,}860 \text{ s} = \mathbf{12.2 \text{ h}}$$

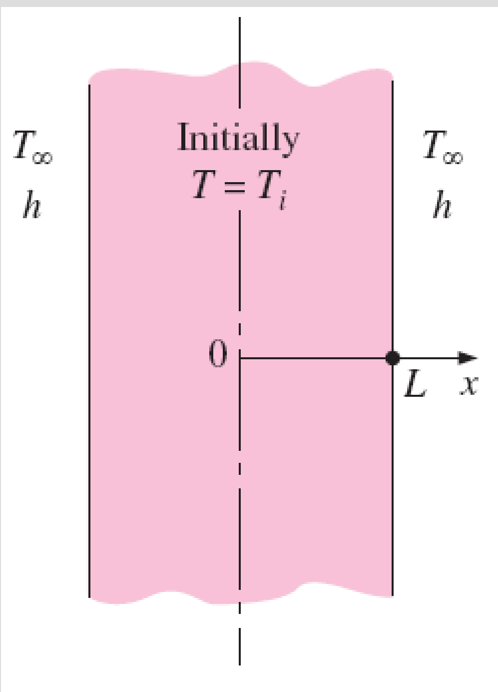
Therefore, as a rough estimate, the person died about 12 h before the body was found, and thus the time of death is 5 AM. This example demonstrates how to obtain "ball park" values using a simple analysis.

TRANSIENT HEAT CONDUCTION IN LARGE PLANE WALLS, LONG CYLINDERS, AND SPHERES WITH SPATIAL EFFECTS

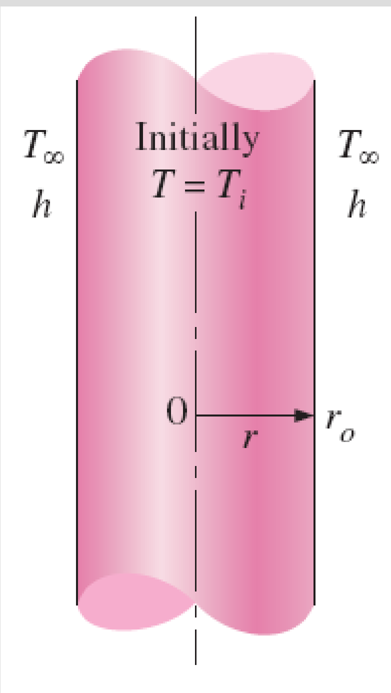
We will consider the variation of temperature with ***time*** and ***position*** in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.



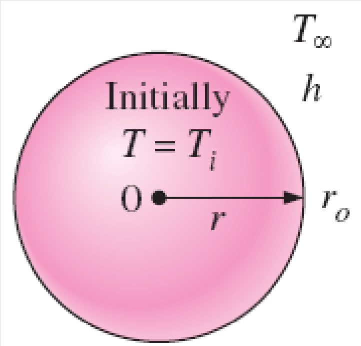
Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$.



(*a*) A large plane wall

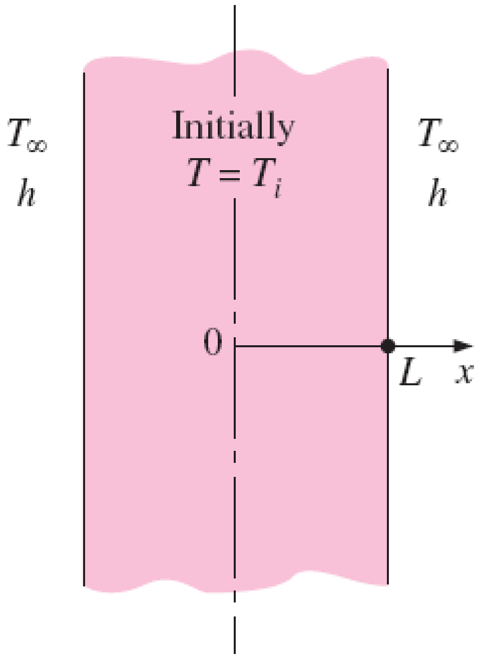


(*b*) A long cylinder



Schematic of the simple geometries in which heat transfer is one-dimensional.

Nondimensionalized One-Dimensional Transient Conduction Problem



(a) A large plane wall

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions:

$$\frac{\partial T(0, t)}{\partial x} = 0 \quad \text{and} \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

Initial condition: $T(x, 0) = T_i$

$\alpha = k/\rho c_p$ $\quad X = x/L$ $\quad \theta(x, t) = [T(x, t) - T_\infty]/[T_i - T_\infty]$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{and} \quad \frac{\partial \theta(1, t)}{\partial X} = \frac{hL}{k}\theta(1, t)$$

Dimensionless differential equation: $$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$$

Dimensionless BC's: $$\frac{\partial \theta(0, \tau)}{\partial X} = 0 \quad \text{and} \quad \frac{\partial \theta(1, \tau)}{\partial X} = -\text{Bi}\,\theta(1, \tau)$$

Dimensionless initial condition: $\theta(X, 0) = 1$

$$\theta(X, \tau) = \frac{T(x, t) - T_i}{T_\infty - T_i}$$ *Dimensionless temperature*

$$X = \frac{x}{L}$$ *Dimensionless distance from the center*

$$\text{Bi} = \frac{hL}{k}$$ *Dimensionless heat transfer coefficient* (Biot number)

$$\tau = \frac{\alpha t}{L^2} = \text{Fo}$$ *Dimensionless time* (Fourier number)

(*a*) Original heat conduction problem:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad T(x, 0) = T_i$$

$$\frac{\partial T(0, t)}{\partial x} = 0, \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

$$T = F(x, L, t, k, \alpha, h, T_i)$$

(*b*) Nondimensionalized problem:

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}, \quad \theta(X, 0) = 1$$

$$\frac{\partial \theta(0, \tau)}{\partial X} = 0, \quad \frac{\partial \theta(1, \tau)}{\partial X} = -\text{Bi}\,\theta(1, \tau)$$

$$\theta = f(X, \text{Bi}, \tau)$$

Nondimensionalization reduces the number of independent variables in one-dimensional transient conduction problems from 8 to 3, offering great convenience in the presentation of results.

TABLE 4–1

Summary of the solutions for one-dimensional transient conduction in a plane wall of thickness $2L$, a cylinder of radius r_o and a sphere of radius r_o subjected to convention from all surfaces.*

Geometry	Solution	λ_n's are the roots of
Plane wall	$\theta = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos(\lambda_n x/L)$	$\mathrm{l_n} \tan \mathrm{l_n}$ 5 Bi
Cylinder	$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \tau} J_0(\lambda_n r/r_o)$	$\lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} = \mathrm{Bi}$
Sphere	$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \frac{\sin(\lambda_n x/L)}{\lambda_n x/L}$	$1 - \lambda_n \cot \lambda_n = \mathrm{Bi}$

*Here $\theta = (T - T_\infty)/(T_i - T_\infty)$ is the dimensionless temperature, $\mathrm{Bi} = hL/k$ or hr_o/k is the Biot number, $\mathrm{Fo} = \tau = \alpha t / L^2$ or $\alpha\tau / r_o^2$ is the Fourier number, and J_0 and J_1 are the Bessel functions of the first kind whose values are given in Table 4–3.

$$\theta_n = A_n\, e^{-\lambda_n^2 \tau} \cos(\lambda_n X)$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

$$\lambda_n \tan \lambda_n = \text{Bi}$$

For Bi = 5, $X = 1$, and $t = 0.2$:

n	λ_n	A_n	θ_n
1	1.3138	1.2402	0.22321
2	4.0336	−0.3442	0.00835
3	6.9096	0.1588	0.00001
4	9.8928	−0.876	0.00000

The analytical solutions of transient conduction problems typically involve infinite series, and thus the evaluation of an infinite number of terms to determine the temperature at a specified location and time.

The term in the series solution of transient conduction problems decline rapidly as n and thus λ_n increases because of the exponential decay function with the exponent $-\lambda_n \tau$.

Approximate Analytical and Graphical Solutions

The terms in the series solutions converge rapidly with increasing time, and for $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent.

Solution with *one-term approximation*

Plane wall: $$\theta_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L), \quad \tau > 0.2$$

Cylinder: $$\theta_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_o), \quad \tau > 0.2$$

Sphere: $$\theta_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}, \quad \tau > 0.2$$

Center of plane wall $(x = 0)$: $$\theta_{0,\text{ wall}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of cylinder $(r = 0)$: $$\theta_{0,\text{ cyl}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Center of sphere $(r = 0)$: $$\theta_{0,\text{ sph}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

TABLE 4–2

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres (Bi = hL/k for a plane wall of thickness $2L$, and Bi = hr_o/k for a cylinder or sphere of radius r_o)

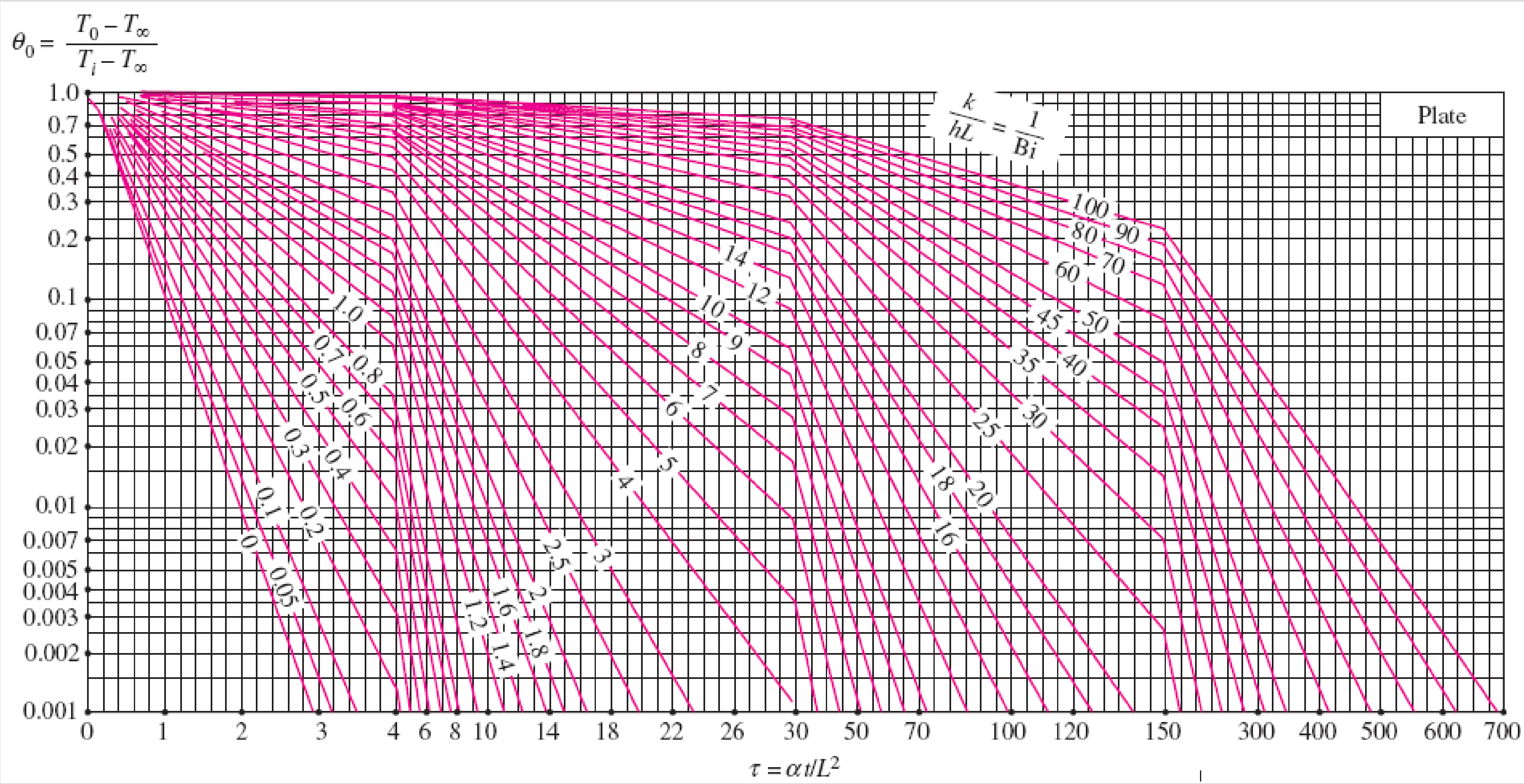
	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

TABLE 4–3

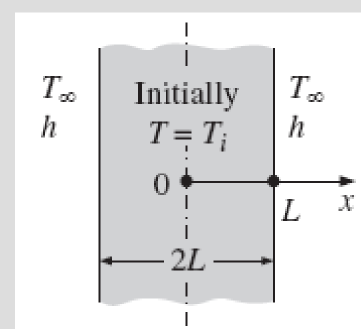
The zeroth- and first-order Bessel functions of the first kind

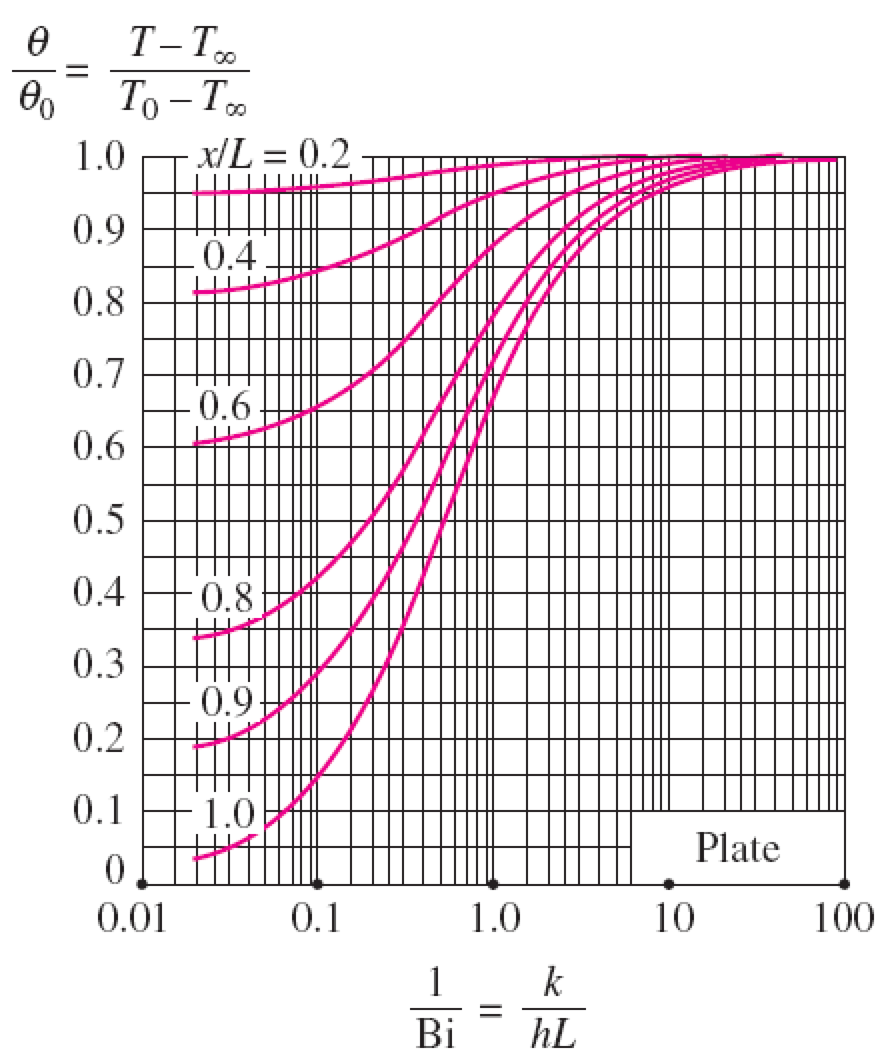
η	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

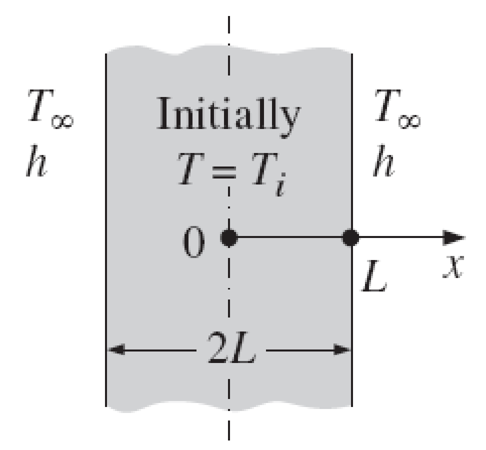
(*a*) Midplane temperature



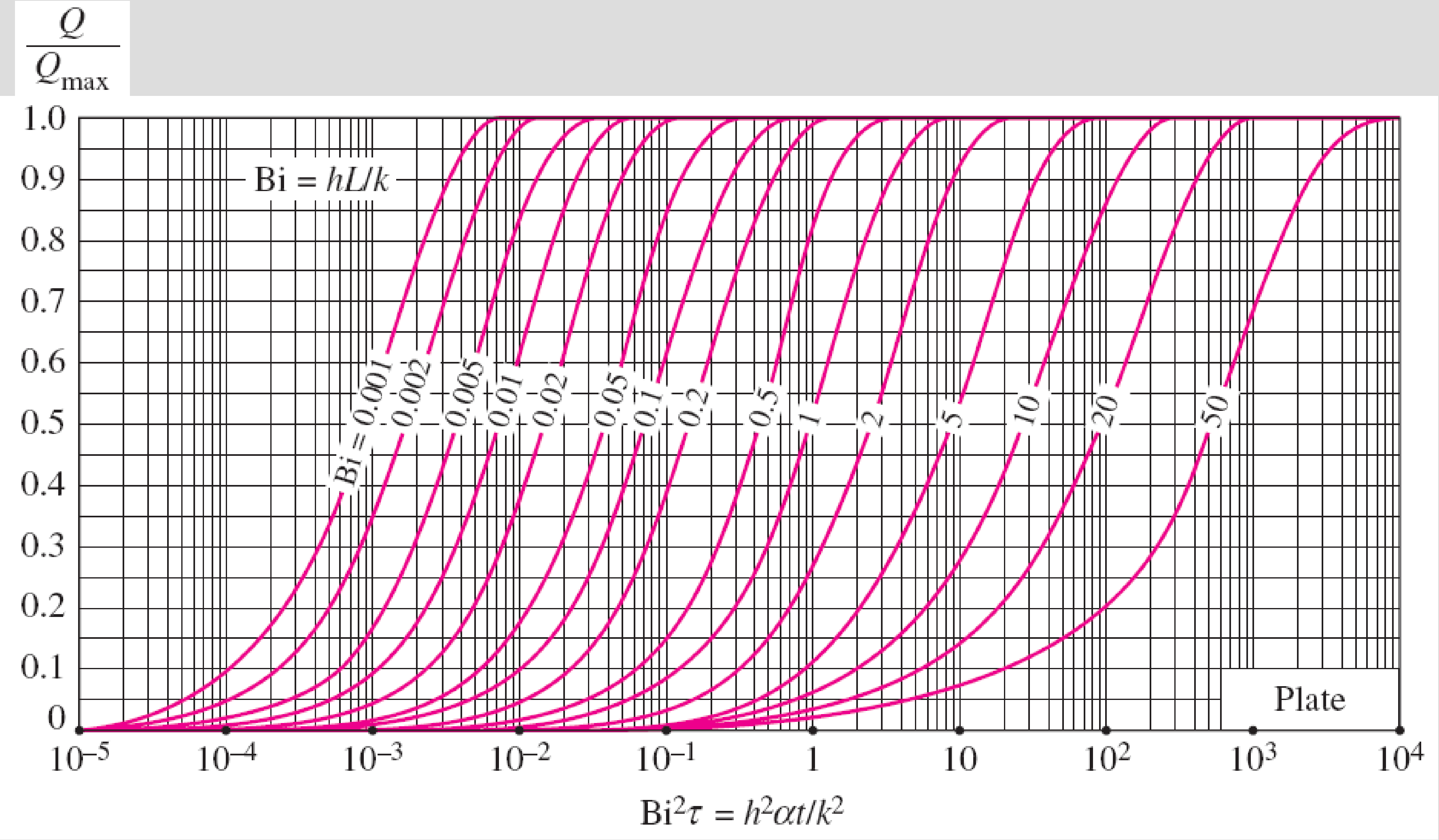
Transient temperature and heat transfer charts (Heisler and Grober charts) for a plane wall of thickness 2*L* initially at a uniform temperature T_i subjected to convection from both sides to an environment at temperature T_∞ with a convection coefficient of *h*.



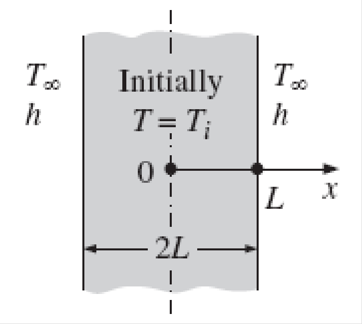




(*b*) Temperature distribution

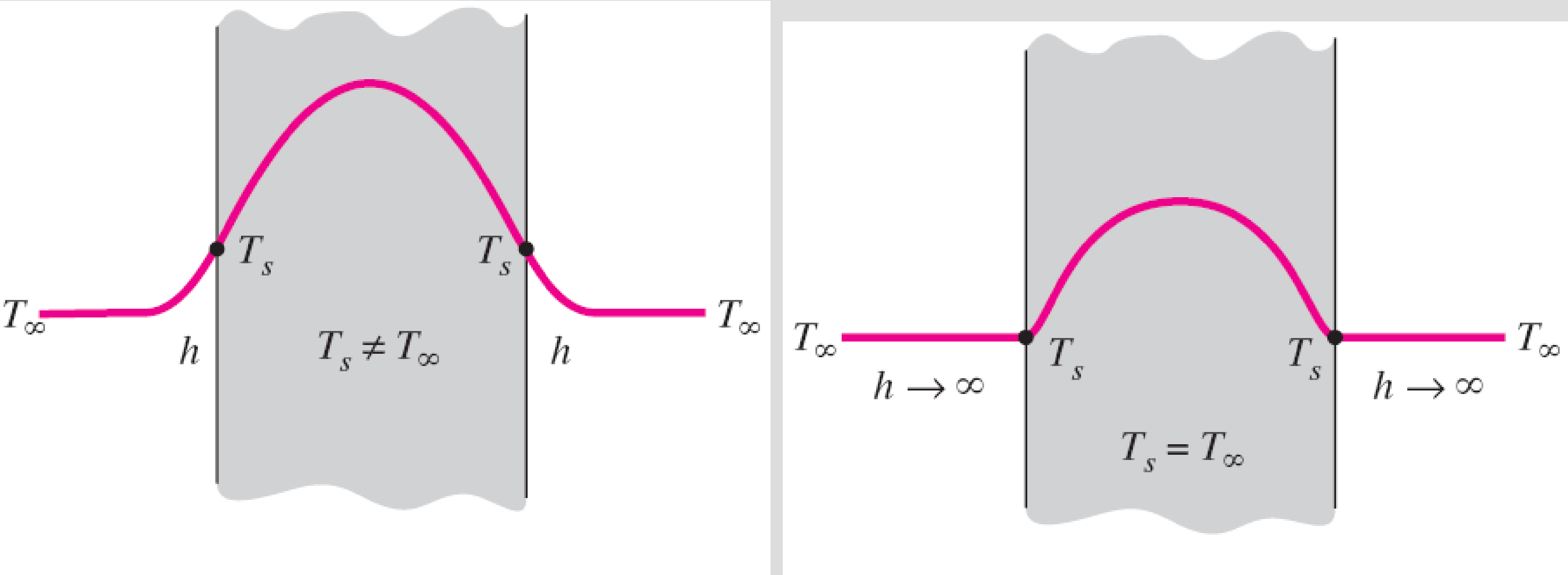


(*c*) Heat transfer



The dimensionless temperatures anywhere in a plane wall, cylinder, and sphere are related to the center temperature by

$$\frac{\theta_{\text{wall}}}{\theta_{0,\text{ wall}}} = \cos\left(\frac{\lambda_1 x}{L}\right), \quad \frac{\theta_{\text{cyl}}}{\theta_{0,\text{ cyl}}} = J_0\left(\frac{\lambda_1 r}{r_o}\right), \quad \text{and} \quad \frac{\theta_{\text{sph}}}{\theta_{0,\text{ sph}}} = \frac{\sin(\lambda_1 r/r_o)}{\lambda_1 r/r_o}$$



(a) Finite convection coefficient (b) Infinite convection coefficient

The specified surface temperature corresponds to the case of convection to an environment at T_∞ with with a convection coefficient h that is *infinite.*

$$Q_{max} = mc_p(T_\infty - T_i) = \rho V c_p(T_\infty - T_i) \quad \text{(kJ)}$$

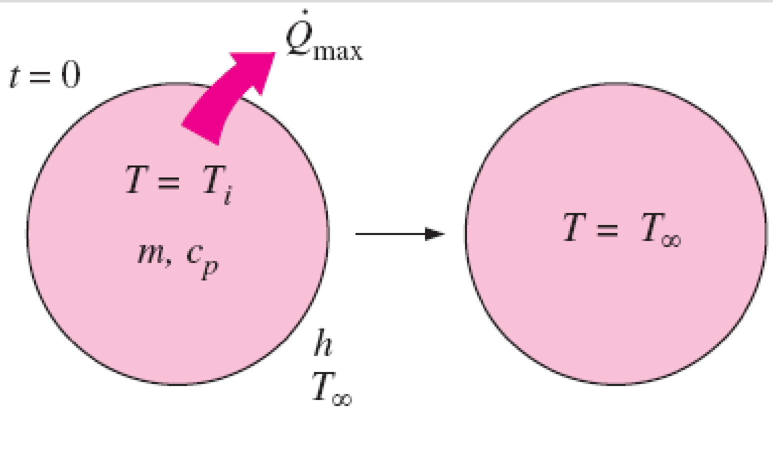
$$\frac{Q}{Q_{max}} = \frac{\int_V \rho c_p[T(x,t) - T_i]dV}{\rho c_p(T_\infty - T_i)V} = \frac{1}{V}\int_V (1-\theta)dV$$

Plane wall:
$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{0,\,wall}\frac{\sin\lambda_1}{\lambda_1}$$

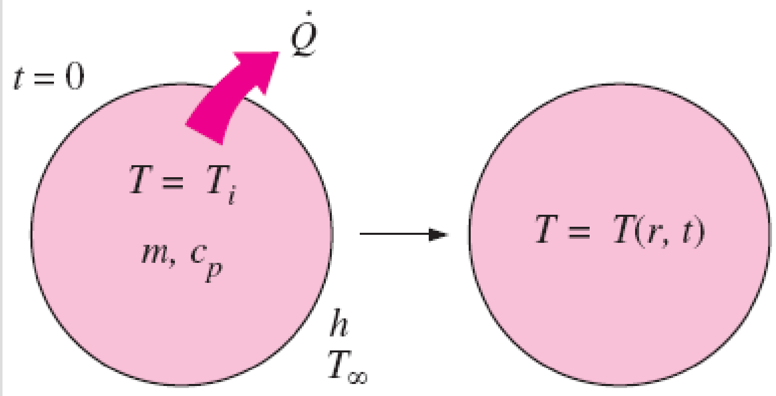
Cylinder:
$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{0,\,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere:
$$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - 3\theta_{0,\,sph}\frac{\sin\lambda_1 - \lambda_1\cos\lambda_1}{\lambda_1^3}$$

$$Q = \int_V \rho c_p[T(x,t) - T_i]dV$$



(*a*) Maximum heat transfer ($t \to \infty$)



$$\left.\begin{aligned} &\text{Bi} = \ldots \\ &\frac{h^2\alpha t}{k^2} = \text{Bi}^2\tau = \ldots \end{aligned}\right\} \quad \frac{Q}{Q_{max}} = \ldots$$

(Gröber chart)

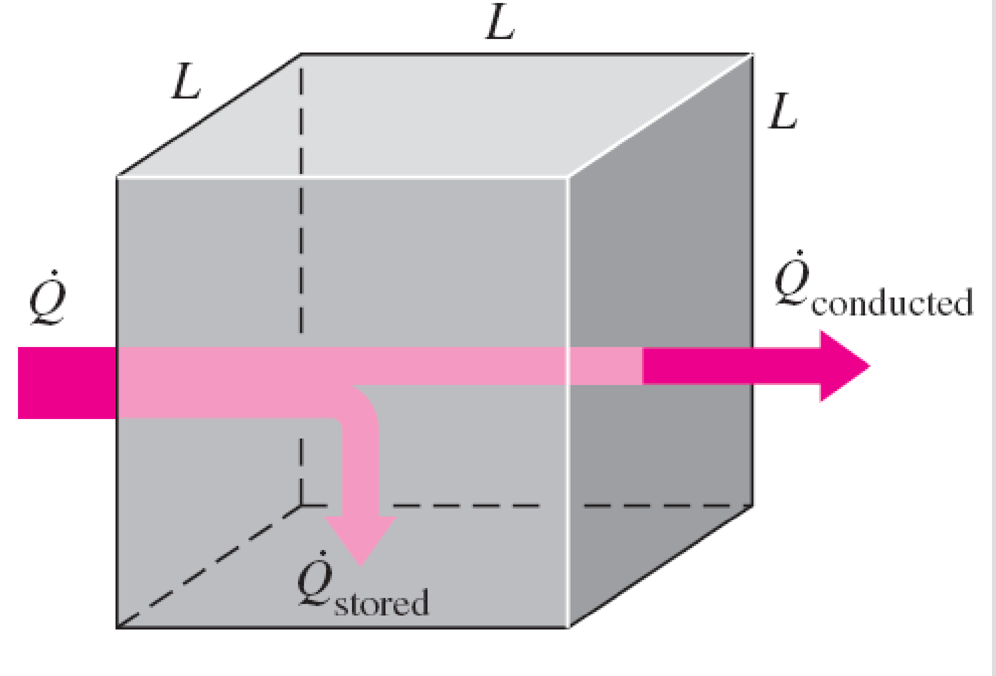
(*b*) Actual heat transfer for time *t*

The fraction of total heat transfer Q/Q_{max} up to a specified time *t* is determined using the Gröber charts.

The physical significance of the *Fourier number*

$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2\,(1/L)}{\rho c_p L^3/t}\frac{\Delta T}{\Delta T} = \frac{\text{The rate at which heat is } \textit{conducted} \text{ across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is } \textit{stored} \text{ in a body of volume } L^3}$$

- The Fourier number is a measure of *heat conducted* through a body relative to *heat stored.*
- A large value of the Fourier number indicates faster propagation of heat through a body.



Fourier number: $\tau = \dfrac{\alpha t}{L^2} = \dfrac{\dot{Q}_{\text{conducted}}}{\dot{Q}_{\text{stored}}}$

Fourier number at time *t* can be viewed as the ratio of the rate of heat conducted to the rate of heat stored at that time.

EXAMPLE 4–3 Boiling Eggs

An ordinary egg can be approximated as a 5-cm-diameter sphere (Fig. 4–19). The egg is initially at a uniform temperature of 5°C and is dropped into boiling water at 95°C. Taking the convection heat transfer coefficient to be $h = 1200\ \text{W/m}^2 \cdot °\text{C}$, determine how long it will take for the center of the egg to reach 70°C.

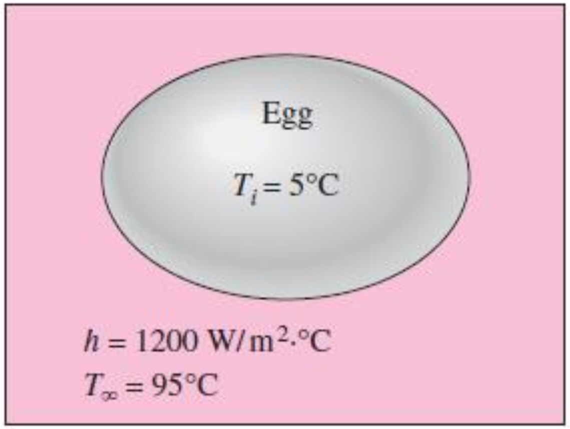


FIGURE 4–19
Schematic for Example 4–3.

SOLUTION An egg is cooked in boiling water. The cooking time of the egg is to be determined.

Assumptions **1** The egg is spherical in shape with a radius of $r_0 = 2.5$ cm. **2** Heat conduction in the egg is one-dimensional because of thermal symmetry about the midpoint. **3** The thermal properties of the egg and the heat transfer coefficient are constant. **4** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The water content of eggs is about 74 percent, and thus the thermal conductivity and diffusivity of eggs can be approximated by those of water at the average temperature of $(5 + 70)/2 = 37.5°C$; $k = 0.627$ W/m · °C and $\alpha = k/\rho C_p = 0.151 \times 10^{-6}$ m²/s (Table A-9).

Analysis The temperature within the egg varies with radial distance as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts or the one-term solutions. Here we will use the latter to demonstrate their use. The Biot number for this problem is

$$\text{Bi} = \frac{hr_0}{k} = \frac{(1200 \text{ W/m}^2 \cdot °\text{C})(0.025 \text{ m})}{0.627 \text{ W/m} \cdot °\text{C}} = 47.8$$

which is much greater than 0.1, and thus the lumped system analysis is not applicable. The coefficients λ_1 and A_1 for a sphere corresponding to this Bi are, from Table 4–1,

$$\lambda_1 = 3.0753, \qquad A_1 = 1.9958$$

Substituting these and other values into Eq. 4–15 and solving for τ gives

$$\frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \longrightarrow \frac{70 - 95}{5 - 95} = 1.9958 e^{-(3.0753)^2 \tau} \longrightarrow \tau = 0.209$$

which is greater than 0.2, and thus the one-term solution is applicable with an error of less than 2 percent. Then the cooking time is determined from the definition of the Fourier number to be

$$t = \frac{\tau r_o^2}{\alpha} = \frac{(0.209)(0.025 \text{ m})^2}{0.151 \times 10^{-6} \text{ m}^2/\text{s}} = 865 \text{ s} \approx \mathbf{14.4 \text{ min}}$$

Therefore, it will take about 15 min for the center of the egg to be heated from 5°C to 70°C.

Discussion Note that the Biot number in lumped system analysis was defined differently as $\text{Bi} = hL_c/k = h(r/3)/k$. However, either definition can be used in determining the applicability of the lumped system analysis unless $\text{Bi} \approx 0.1$.

EXAMPLE 4–4 Heating of Large Brass Plates in an Oven

In a production facility, large brass plates of 4 cm thickness that are initially at a uniform temperature of 20°C are heated by passing them through an oven that is maintained at 500°C (Fig. 4–20). The plates remain in the oven for a period of 7 min. Taking the combined convection and radiation heat transfer coefficient to be $h = 120\ \text{W/m}^2\cdot{}^\circ\text{C}$, determine the surface temperature of the plates when they come out of the oven.

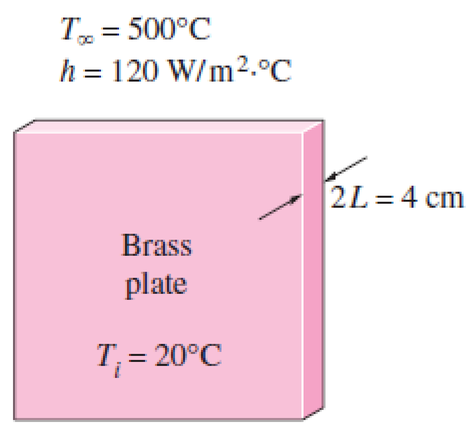


FIGURE 4–20
Schematic for Example 4–4.

SOLUTION Large brass plates are heated in an oven. The surface temperature of the plates leaving the oven is to be determined.

Assumptions **1** Heat conduction in the plate is one-dimensional since the plate is large relative to its thickness and there is thermal symmetry about the center plane. **2** The thermal properties of the plate and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of brass at room temperature are $k = 110$ W/m · °C, $\rho = 8530$ kg/m^3, $C_p = 380$ J/kg · °C, and $\alpha = 33.9 \times 10^{-6}$ m^2/s (Table A-3). More accurate results are obtained by using properties at average temperature.

Analysis The temperature at a specified location at a given time can be determined from the Heisler charts or one-term solutions. Here we will use the charts to demonstrate their use. Noting that the half-thickness of the plate is $L = 0.02$ m, from Fig. 4–13 we have

$$\left.\begin{aligned} \frac{1}{\text{Bi}} &= \frac{k}{hL} = \frac{100 \text{ W/m} \cdot \text{°C}}{(120 \text{ W/m}^2 \cdot \text{°C})(0.02 \text{ m})} = 45.8 \\ \tau &= \frac{\alpha t}{L^2} = \frac{(33.9 \times 10^{-6} \text{ m}^2\text{/s})(7 \times 60 \text{ s})}{(0.02 \text{ m})^2} = 35.6 \end{aligned}\right\} \frac{T_o - T_\infty}{T_i - T_\infty} = 0.46$$

Also,

$$\left.\begin{aligned} \frac{1}{\text{Bi}} &= \frac{k}{hL} = 45.8 \\ \frac{x}{L} &= \frac{L}{L} = 1 \end{aligned}\right\} \frac{T - T_\infty}{T_o - T_\infty} = 0.99$$

Therefore,

$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{T - T_\infty}{T_o - T_\infty}\frac{T_o - T_\infty}{T_i - T_\infty} = 0.46 \times 0.99 = 0.455$$

and

$$T = T_\infty + 0.455(T_i - T_\infty) = 500 + 0.455(20 - 500) = 282\text{°C}$$

Therefore, the surface temperature of the plates will be 282°C when they leave the oven.

Discussion We notice that the Biot number in this case is Bi = 1/45.8 = 0.022, which is much less than 0.1. Therefore, we expect the lumped system analysis to be applicable. This is also evident from $(T - T_\infty)/(T_o - T_\infty) = 0.99$, which indicates that the temperatures at the center and the surface of the plate relative to the surrounding temperature are within 1 percent of each other.

Noting that the error involved in reading the Heisler charts is typically at least a few percent, the lumped system analysis in this case may yield just as accurate results with less effort.

The heat transfer surface area of the plate is $2A$, where A is the face area of the plate (the plate transfers heat through both of its surfaces), and the volume of the plate is $V = (2L)A$, where L is the half-thickness of the plate. The exponent b used in the lumped system analysis is determined to be

$$b = \frac{hA_s}{\rho C_p V} = \frac{h(2A)}{\rho C_p (2LA)} = \frac{h}{\rho C_p L}$$

$$= \frac{120 \text{ W/m}^2 \cdot {}^\circ\text{C}}{(8530 \text{ kg/m}^3)(380 \text{ J/kg} \cdot {}^\circ\text{C})(0.02 \text{ m})} = 0.00185 \text{ s}^{-1}$$

Then the temperature of the plate at $t = 7$ min $= 420$ s is determined from

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \longrightarrow \frac{T(t) - 500}{20 - 500} = e^{-(0.00185\ s^{-1})(420\ \text{s})}$$

It yields

$$T(t) = 279^\circ\text{C}$$

which is practically identical to the result obtained above using the Heisler charts. Therefore, we can use lumped system analysis with confidence when the Biot number is sufficiently small.

EXAMPLE 4–5 **Cooling of a Long Stainless Steel Cylindrical Shaft**

A long 20-cm-diameter cylindrical shaft made of stainless steel 304 comes out of an oven at a uniform temperature of 600°C (Fig. 4–21). The shaft is then allowed to cool slowly in an environment chamber at 200°C with an average heat transfer coefficient of $h = 80\ \text{W/m}^2 \cdot °\text{C}$. Determine the temperature at the center of the shaft 45 min after the start of the cooling process. Also, determine the heat transfer per unit length of the shaft during this time period.

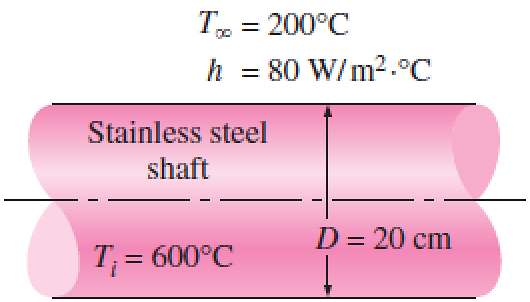


FIGURE 4–21
Schematic for Example 4–5.

SOLUTION A long cylindrical shaft at 600°C is allowed to cool slowly. The center temperature and the heat transfer per unit length are to be determined.

Assumptions **1** Heat conduction in the shaft is one-dimensional since it is long and it has thermal symmetry about the centerline. **2** The thermal properties of the shaft and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of stainless steel 304 at room temperature are $k = 14.9$ W/m · °C, $\rho = 7900$ kg/m^3, $C_p = 477$ J/kg · °C, and $\alpha = 3.95 \times 10^{-6}$ m^2/s (Table A-3). More accurate results can be obtained by using properties at average temperature.

Analysis The temperature within the shaft may vary with the radial distance r as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts. Noting that the radius of the shaft is $r_o = 0.1$ m, from Fig. 4–14 we have

$$\left.\begin{aligned} \frac{1}{\text{Bi}} &= \frac{k}{hr_o} = \frac{14.9 \text{ W/m} \cdot {}^\circ\text{C}}{(80 \text{ W/m}^2 \cdot {}^\circ\text{C})(0.1 \text{ m})} = 1.86 \\ \tau &= \frac{\alpha t}{r_o^2} = \frac{(3.95 \times 10^{-6} \text{ m}^2\text{/s})(45 \times 60 \text{ s})}{(0.1 \text{ m})^2} = 1.07 \end{aligned}\right\} \frac{T_o - T_\infty}{T_i - T_\infty} = 0.40$$

and

$$T_o = T_\infty + 0.4(T_i - T_\infty) = 200 + 0.4(600 - 200) = \mathbf{360^\circ C}$$

Therefore, the center temperature of the shaft will drop from 600°C to 360°C in 45 min.

To determine the actual heat transfer, we first need to calculate the maximum heat that can be transferred from the cylinder, which is the sensible energy of the cylinder relative to its environment. Taking $L = 1$ m,

$$\begin{aligned} m &= \rho V = \rho \pi r_o^2 L = (7900 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(1 \text{ m}) = 248.2 \text{ kg} \\ Q_{\max} &= mC_p(T_\infty - T_i) = (248.2 \text{ kg})(0.477 \text{ kJ/kg} \cdot {}^\circ\text{C})(600 - 200)^\circ\text{C} \\ &= 47{,}354 \text{ kJ} \end{aligned}$$

The dimensionless heat transfer ratio is determined from Fig. 4–14*c* for a long cylinder to be

$$\left.\begin{aligned} \text{Bi} &= \frac{1}{1/\text{Bi}} = \frac{1}{1.86} = 0.537 \\ \frac{h^2 \alpha t}{k^2} &= \text{Bi}^2\tau = (0.537)^2(1.07) = 0.309 \end{aligned}\right\} \frac{Q}{Q_{\max}} = 0.62$$

Therefore,

$$Q = 0.62 Q_{\max} = 0.62 \times (47{,}354 \text{ kJ}) = \mathbf{29{,}360 \text{ kJ}}$$

which is the total heat transfer from the shaft during the first 45 min of the cooling.

ALTERNATIVE SOLUTION We could also solve this problem using the one-term solution relation instead of the transient charts. First we find the Biot number

$$\text{Bi} = \frac{hr_o}{k} = \frac{(80 \text{ W/m}^2 \cdot {}^\circ\text{C})(0.1 \text{ m})}{14.9 \text{ W/m} \cdot {}^\circ\text{C}} = 0.537$$

The coefficients λ_1 and A_1 for a cylinder corresponding to this Bi are determined from Table 4–1 to be

$$\lambda_1 = 0.970, \qquad A_1 = 1.122$$

Substituting these values into Eq. 4–14 gives

$$\theta_0 = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} = 1.122 e^{-(0.970)^2(1.07)} = 0.41$$

and thus

$$T_o = T_\infty + 0.41(T_i - T_\infty) = 200 + 0.41(600 - 200) = \mathbf{364°C}$$

The value of $J_1(\lambda_1)$ for $\lambda_1 = 0.970$ is determined from Table 4–2 to be 0.430. Then the fractional heat transfer is determined from Eq. 4–18 to be

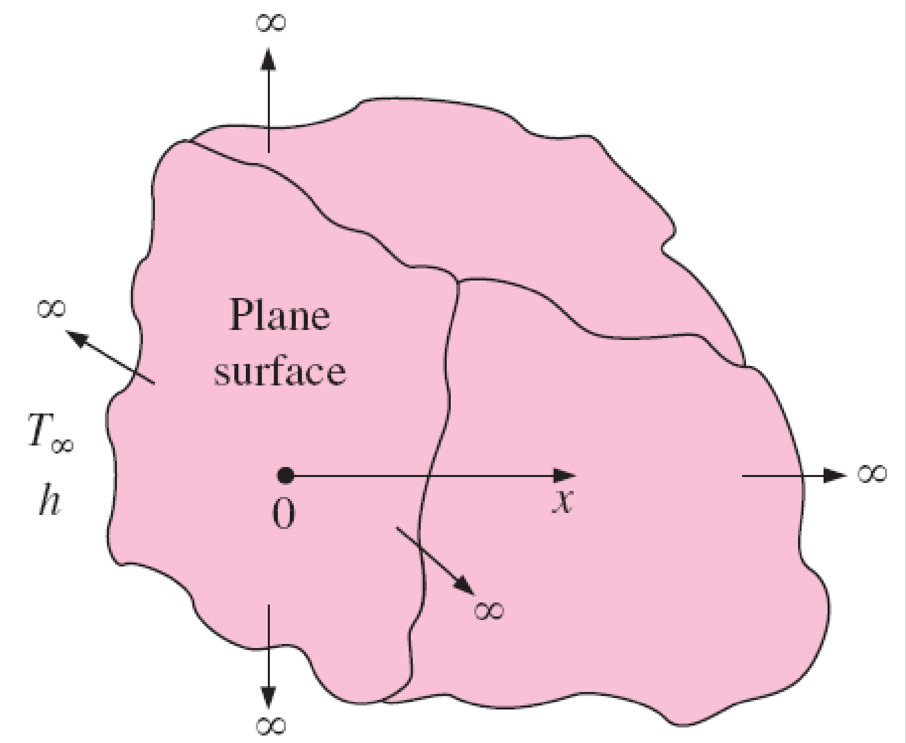
$$\frac{Q}{Q_{max}} = 1 - 2\theta_0 \frac{J_1(\lambda_1)}{\lambda_1} = 1 - 2 \times 0.41 \frac{0.430}{0.970} = 0.636$$

and thus

$$Q = 0.636 Q_{max} = 0.636 \times (47{,}354 \text{ kJ}) = \mathbf{30{,}120 \text{ kJ}}$$

Discussion The slight difference between the two results is due to the reading error of the charts.

TRANSIENT HEAT CONDUCTION IN SEMI-INFINITE SOLIDS



Schematic of a semi-infinite body.

For short periods of time, most bodies can be modeled as semi-infinite solids since heat does not have sufficient time to penetrate deep into the body.

Semi-infinite solid: An idealized body that has a *single plane surface* and extends to infinity in all directions.

The earth can be considered to be a semi-infinite medium in determining the variation of temperature near its surface.

A thick wall can be modeled as a semi-infinite medium if all we are interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

Analytical solution for the case of constant temperature T_s on the surface

Differential equation: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions: $$T(0, t) = T_s \quad \text{and} \quad T(x \to \infty, t) = T_i$$

Initial condition: $$T(x, 0) = T_i$$

Similarity variable: $$\eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{d^2T}{d\eta^2} = -2\eta\frac{dT}{d\eta}$$

$$T(0) = T_s \quad \text{and} \quad T(\eta \to \infty) = T_i$$

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du = \text{erf}(\eta) = 1 - \text{erfc}(\eta)$$

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}du$$

error function

$$\text{erfc}(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta \varepsilon^{-u^2}du$$

complementary error function

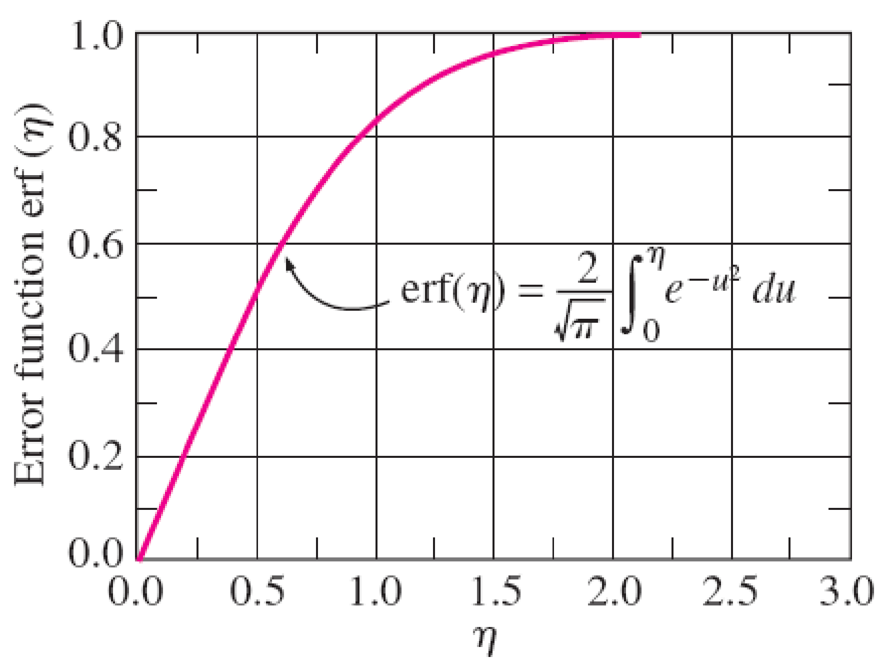
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{and} \quad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = \frac{x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2T}{d\eta^2}$$

Transformation of variables in the derivatives of the heat conduction equation by the use of chain rule.



Error function is a standard mathematical function, just like the sine and cosine functions, whose value varies between 0 and 1.

TABLE 4–4

The complementary error function

η	erfc (η)	η	erfc (η)	η	erfc (η)
0.00	1.00000	0.38	0.5910	0.76	0.2825
0.02	0.9774	0.40	0.5716	0.78	0.2700
0.04	0.9549	0.42	0.5525	0.80	0.2579
0.06	0.9324	0.44	0.5338	0.82	0.2462
0.08	0.9099	0.46	0.5153	0.84	0.2349
0.10	0.8875	0.48	0.4973	0.86	0.2239
0.12	0.8652	0.50	0.4795	0.88	0.2133
0.14	0.8431	0.52	0.4621	0.90	0.2031
0.16	0.8210	0.54	0.4451	0.92	0.1932
0.18	0.7991	0.56	0.4284	0.94	0.1837
0.20	0.7773	0.58	0.4121	0.96	0.1746
0.22	0.7557	0.60	0.3961	0.98	0.1658
0.24	0.7343	0.62	0.3806	1.00	0.1573
0.26	0.7131	0.64	0.3654	1.02	0.1492
0.28	0.6921	0.66	0.3506	1.04	0.1413
0.30	0.6714	0.68	0.3362	1.06	0.1339
0.32	0.6509	0.70	0.3222	1.08	0.1267
0.34	0.6306	0.72	0.3086	1.10	0.1198
0.36	0.6107	0.74	0.2953	1.12	0.1132

$$\dot{q}_s = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = -k\frac{dT}{d\eta}\frac{\partial \eta}{\partial x}\bigg|_{\eta=0} = -kC_1 e^{-\eta^2}\frac{1}{\sqrt{4\alpha t}}\bigg|_{\eta=0} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Analytical solutions for different boundary conditions on the surface

Case 1: Specified Surface Temperature, T_s = constant

$$\frac{T(x,t)-T_i}{T_s-T_i}=\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{and} \quad \dot{q}_s(t)=\frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}}$$

Case 2: Specified Surface Heat Flux, $\dot{q}_s$ = constant.

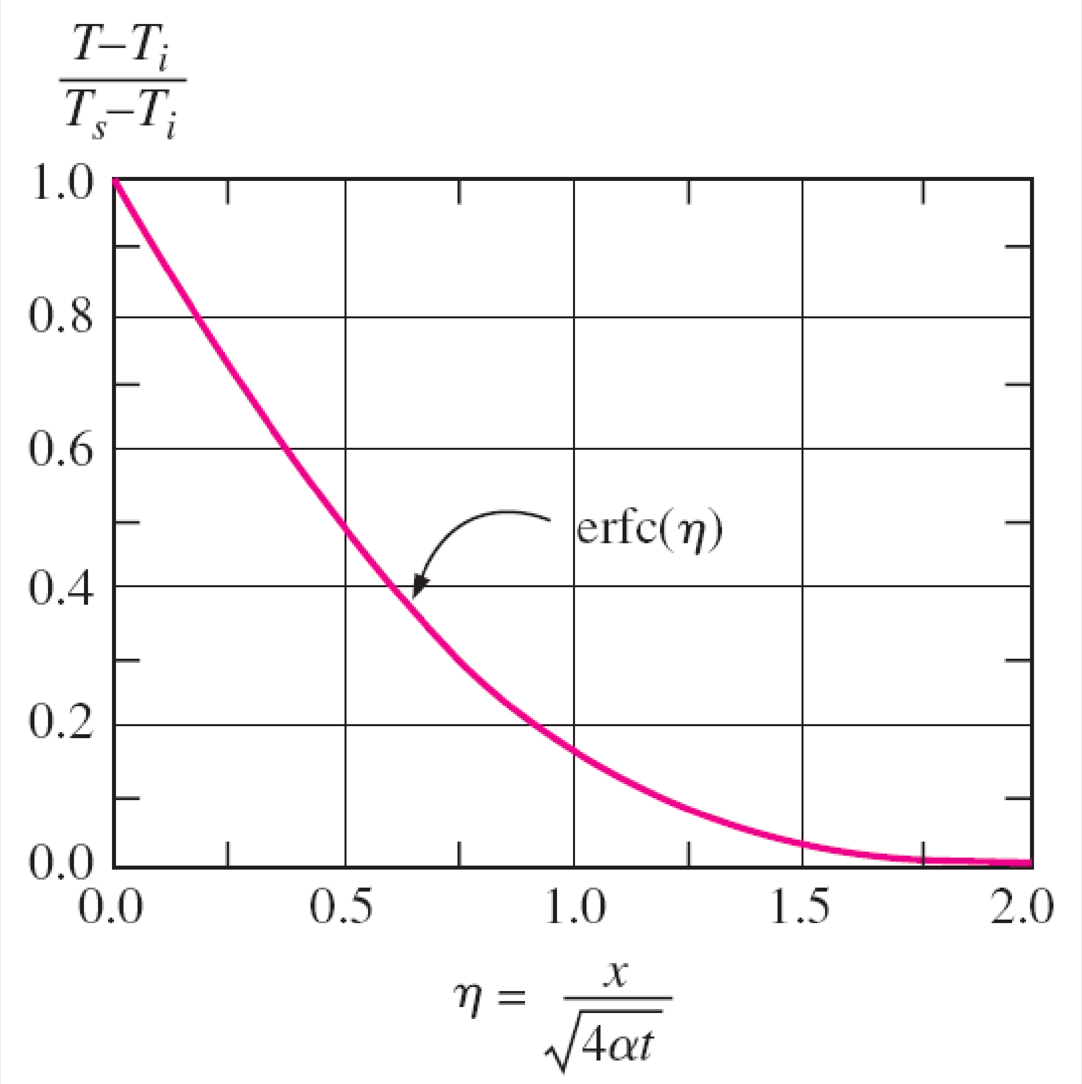
$$T(x,t)-T_i=\frac{\dot{q}_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right)-x\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

Case 3: Convection on the Surface, $\dot{q}_s(t)=h[T_\infty-T(0,t)]$.

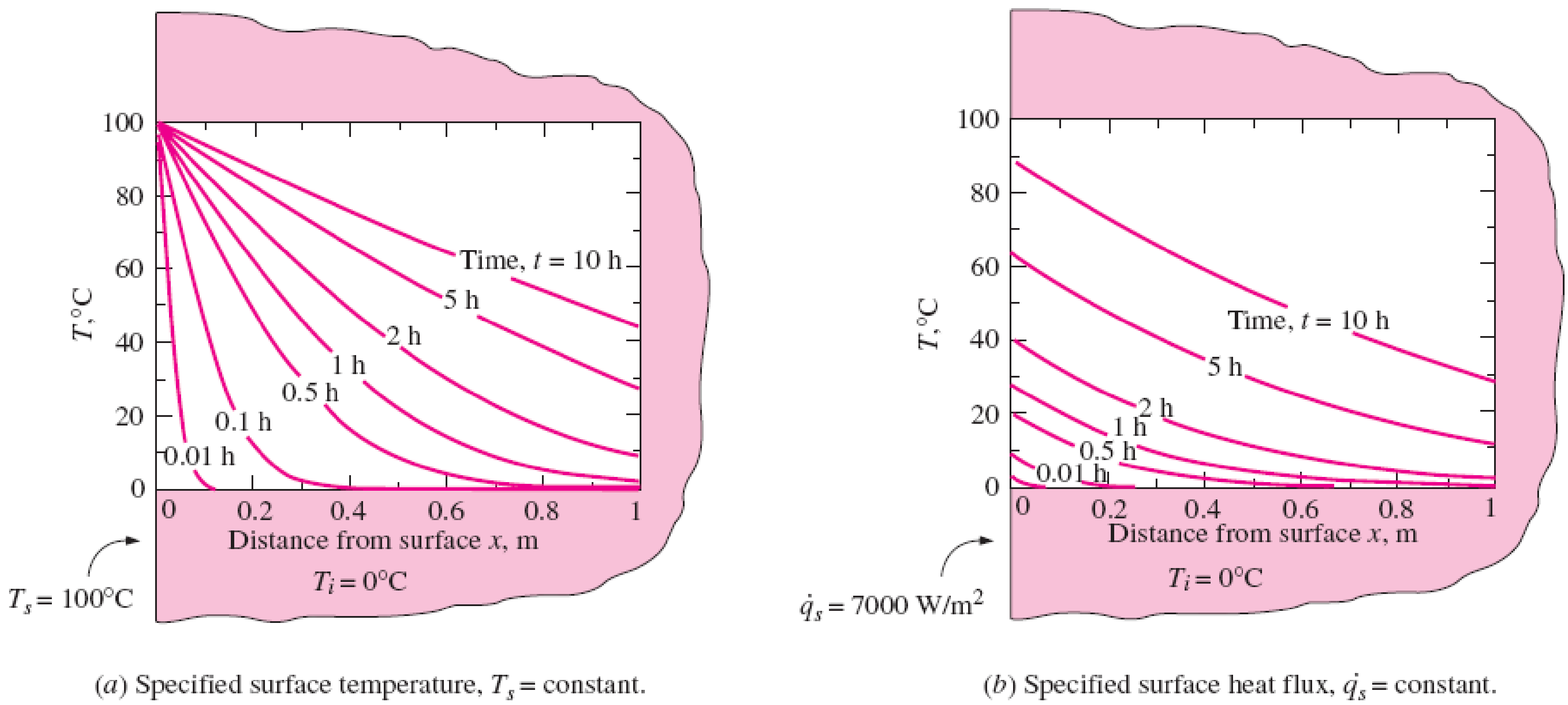
$$\frac{T(x,t)-T_i}{T_\infty-T_i}=\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)-\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)$$

Case 4: Energy Pulse at Surface, e_s = constant.

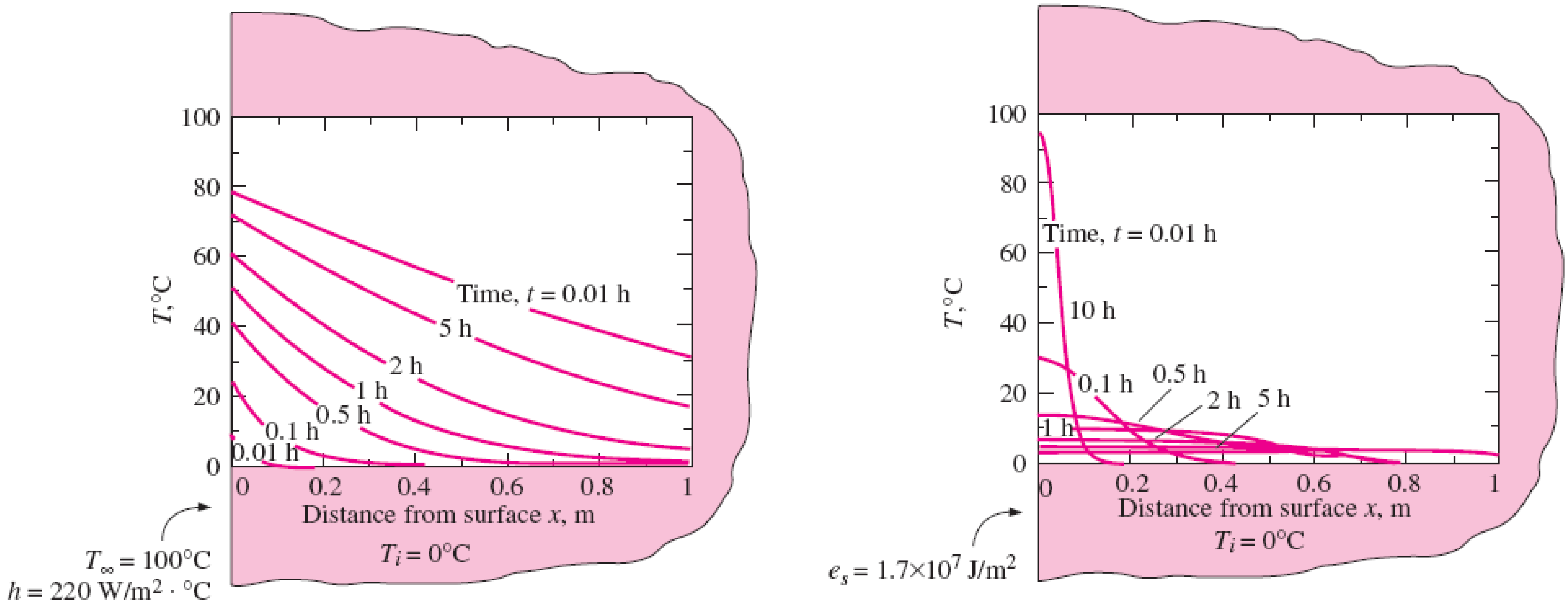
$$T(x,t)-T_i=\frac{e_s}{k\sqrt{\pi t/\alpha}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$



Dimensionless temperature distribution for transient conduction in a semi-infinite solid whose surface is maintained at a constant temperature T_s.



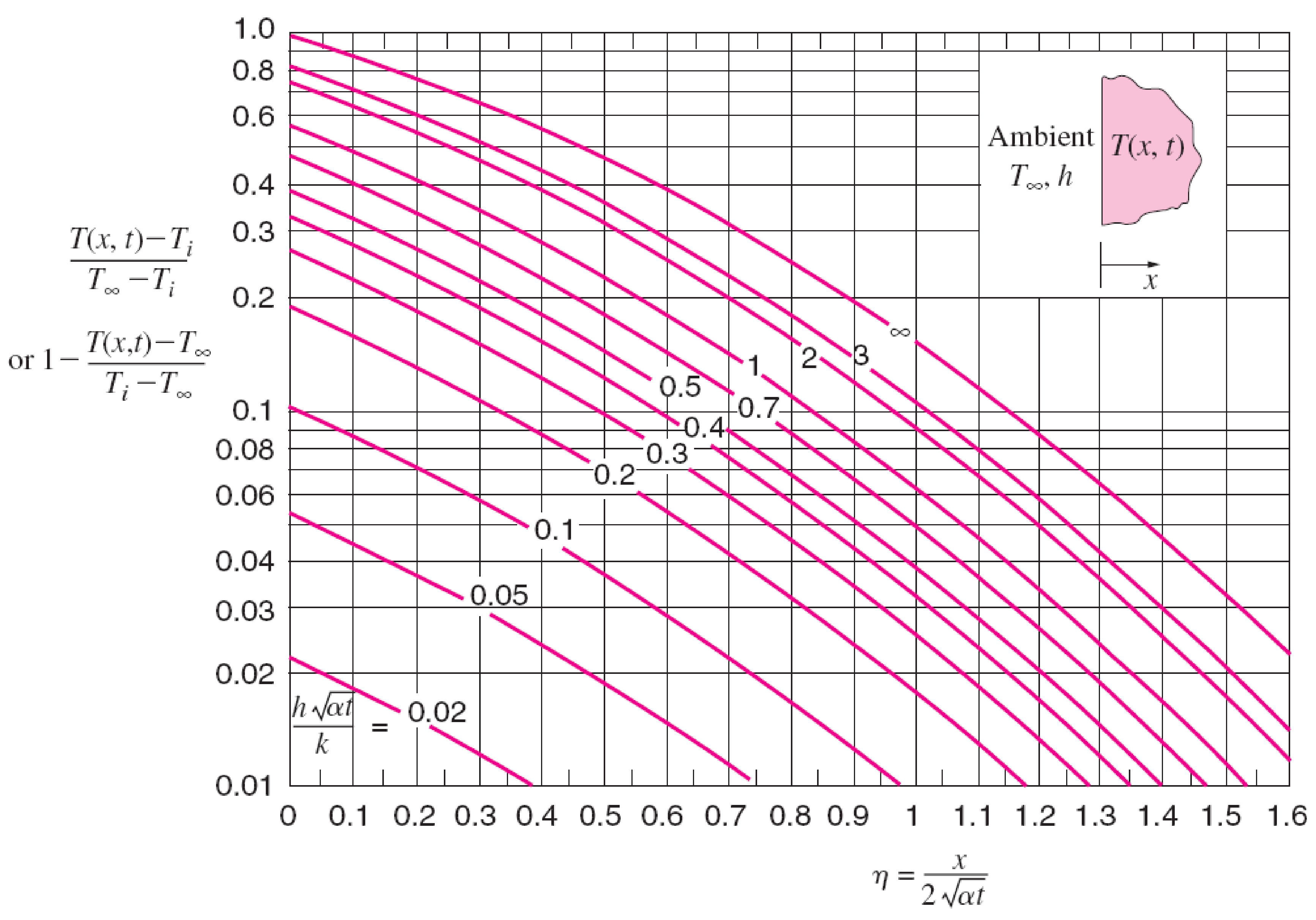
Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}\ m^2/s$, $k = 80.2\ W/m \cdot °C$) initially at 0 °C under different thermal conditions on the surface.



(*c*) Convection at the surface

(*d*) Energy pulse at the surface, e_s = constant

Variations of temperature with position and time in a large cast iron block ($\alpha = 2.31 \times 10^{-5}$ m²/s, $k = 80.2$ W/m · °C) initially at 0 °C under different thermal conditions on the surface.



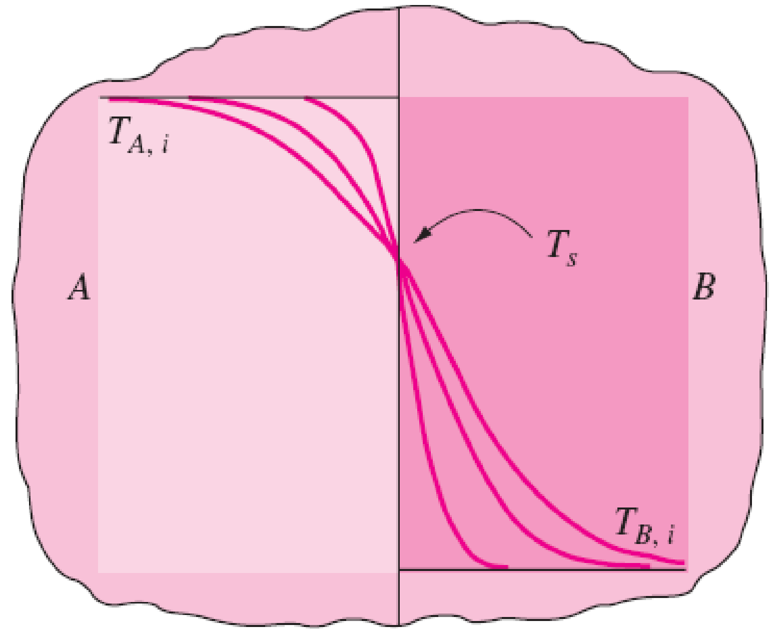
Variation of temperature with position and time in a semi-infinite solid initially at temperature T_i subjected to convection to an environment at T_∞ with a convection heat transfer coefficient of h.

Contact of Two Semi-Infinite Solids

When two large bodies *A* and *B*, initially at uniform temperatures $T_{A,i}$ and $T_{B,i}$ are brought into contact, they instantly achieve temperature equality at the contact surface.

If the two bodies are of the same material, the contact surface temperature is the arithmetic average, $T_s = (T_{A,i}+T_{B,i})/2$.

If the bodies are of different materials, the surface temperature T_s will be different than the arithmetic average.



Contact of two semi-infinite solids of different initial temperatures.

$$\dot{q}_{s,A} = \dot{q}_{s,B} \rightarrow -\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi\alpha_A t}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi\alpha_B t}} \rightarrow \frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \sqrt{\frac{(k\rho c_p)_B}{(k\rho c_p)_A}}$$

$$T_s = \frac{\sqrt{(k\rho c_p)_A}T_{A,i} + \sqrt{(k\rho c_p)_B}T_{B,i}}{\sqrt{(k\rho c_p)_A} + \sqrt{(k\rho c_p)_B}}$$

The interface temperature of two bodies brought into contact is dominated by the body with the larger $k\rho c_p$.

EXAMPLE: When a person with a skin temperature of 35°C touches an aluminum block and then a wood block both at 15°C, the contact surface temperature will be 15.9°C in the case of aluminum and 30°C in the case of wood.

EXAMPLE 4–6 **Minimum Burial Depth of Water Pipes to Avoid Freezing**

In areas where the air temperature remains below 0°C for prolonged periods of time, the freezing of water in underground pipes is a major concern. Fortunately, the soil remains relatively warm during those periods, and it takes weeks for the subfreezing temperatures to reach the water mains in the ground. Thus, the soil effectively serves as an insulation to protect the water from subfreezing temperatures in winter.

The ground at a particular location is covered with snow pack at −10°C for a continuous period of three months, and the average soil properties at that location are $k = 0.4$ W/m · °C and $\alpha = 0.15 \times 10^{-6}$ m²/s (Fig. 4–24). Assuming an initial uniform temperature of 15°C for the ground, determine the minimum burial depth to prevent the water pipes from freezing.

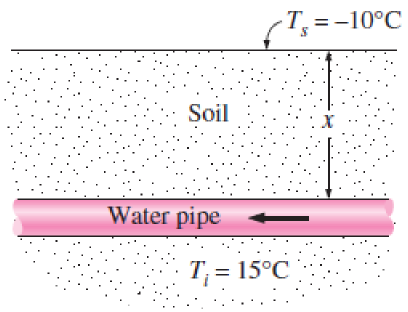


FIGURE 4–24
Schematic for Example 4–6.

SOLUTION The water pipes are buried in the ground to prevent freezing. The minimum burial depth at a particular location is to be determined.

Assumptions **1** The temperature in the soil is affected by the thermal conditions at one surface only, and thus the soil can be considered to be a semi-infinite medium with a specified surface temperature of $-10°C$. **2** The thermal properties of the soil are constant.

Properties The properties of the soil are as given in the problem statement.

Analysis The temperature of the soil surrounding the pipes will be 0°C after three months in the case of minimum burial depth. Therefore, from Fig. 4–23, we have

$$\left.\begin{aligned} &\frac{h\sqrt{\alpha t}}{k} = \infty \qquad \text{(since } h \to \infty\text{)} \\ &1 - \frac{T(x,t) - T_\infty}{T_i - T_\infty} = 1 - \frac{0 - (-10)}{15 - (-10)} = 0.6 \end{aligned}\right\} \xi = \frac{x}{2\sqrt{\alpha t}} = 0.36$$

We note that

$$t = (90 \text{ days})(24 \text{ h/day})(3600 \text{ s/h}) = 7.78 \times 10^6 \text{ s}$$

and thus

$$x = 2\xi\sqrt{\alpha t} = 2 \times 0.36\sqrt{(0.15 \times 10^{-6} \text{ m}^2\text{/s})(7.78 \times 10^6 \text{ s})} = \mathbf{0.77\ m}$$

Therefore, the water pipes must be buried to a depth of at least 77 cm to avoid freezing under the specified harsh winter conditions.

ALTERNATIVE SOLUTION The solution of this problem could also be determined from Eq. 4–24:

$$\frac{T(x,t) - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \longrightarrow \frac{0 - 15}{-10 - 15} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = 0.60$$

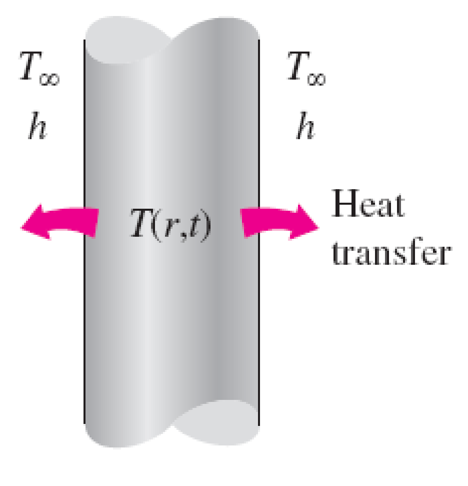
The argument that corresponds to this value of the complementary error function is determined from Table 4–3 to be $\xi = 0.37$. Therefore,

$$x = 2\xi\sqrt{\alpha t} = 2 \times 0.37\sqrt{(0.15 \times 10^{-6} \text{ m}^2\text{/s})(7.78 \times 10^6 \text{ s})} = \mathbf{0.80\ m}$$

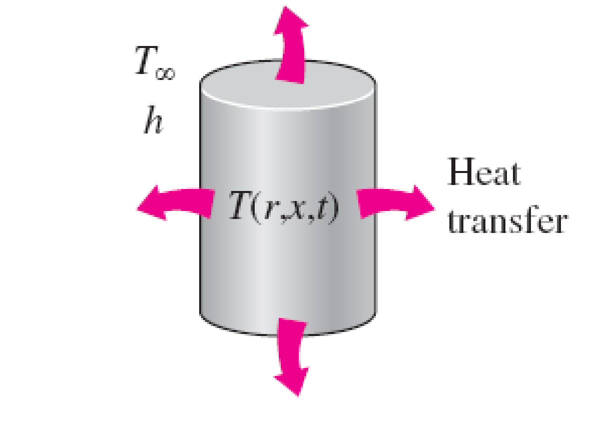
Again, the slight difference is due to the reading error of the chart.

TRANSIENT HEAT CONDUCTION IN MULTIDIMENSIONAL SYSTEMS

- Using a superposition approach called the **product solution**, the transient temperature charts and solutions can be used to construct solutions for the *two-dimensional* and *three-dimensional* transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, a rectangular prism or a semi-infinite rectangular bar, provided that *all* surfaces of the solid are subjected to convection to the *same* fluid at temperature T_∞, with the *same* heat transfer coefficient h, and the body involves no heat generation.
- The solution in such multidimensional geometries can be expressed as the ***product*** of the solutions for the one-dimensional geometries whose intersection is the multidimensional geometry.



(*a*) Long cylinder



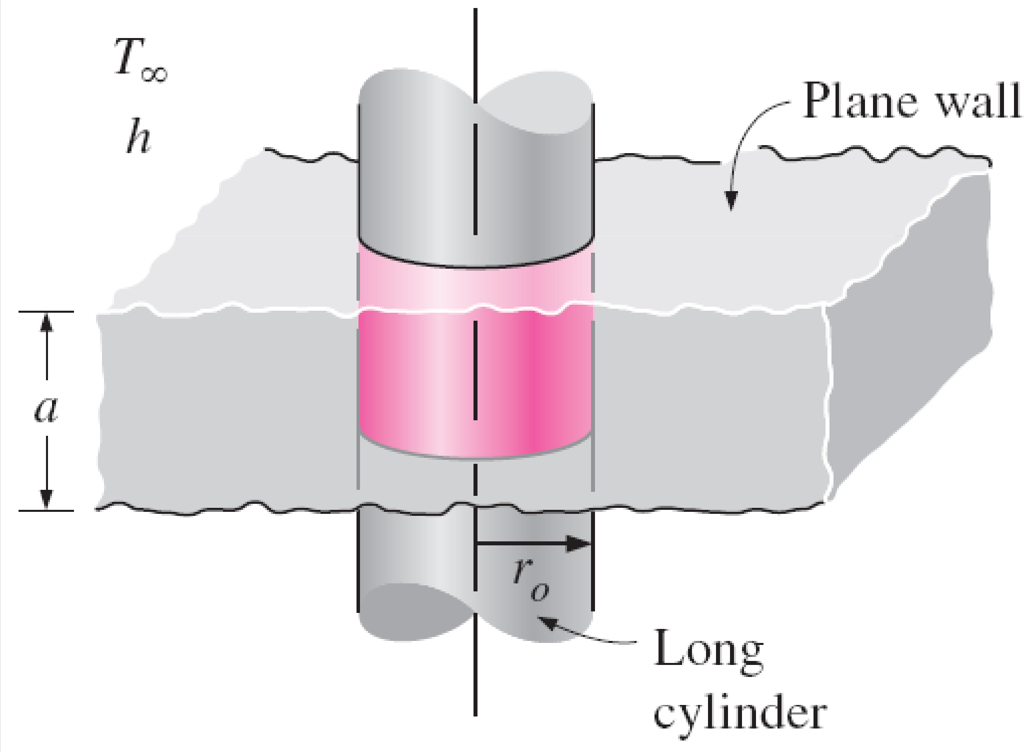
(*b*) Short cylinder (two-dimensional)

The temperature in a short cylinder exposed to convection from all surfaces varies in both the radial and axial directions, and thus heat is transferred in both directions.

The solution for a multidimensional geometry is the product of the solutions of the one-dimensional geometries whose intersection is the multidimensional body.

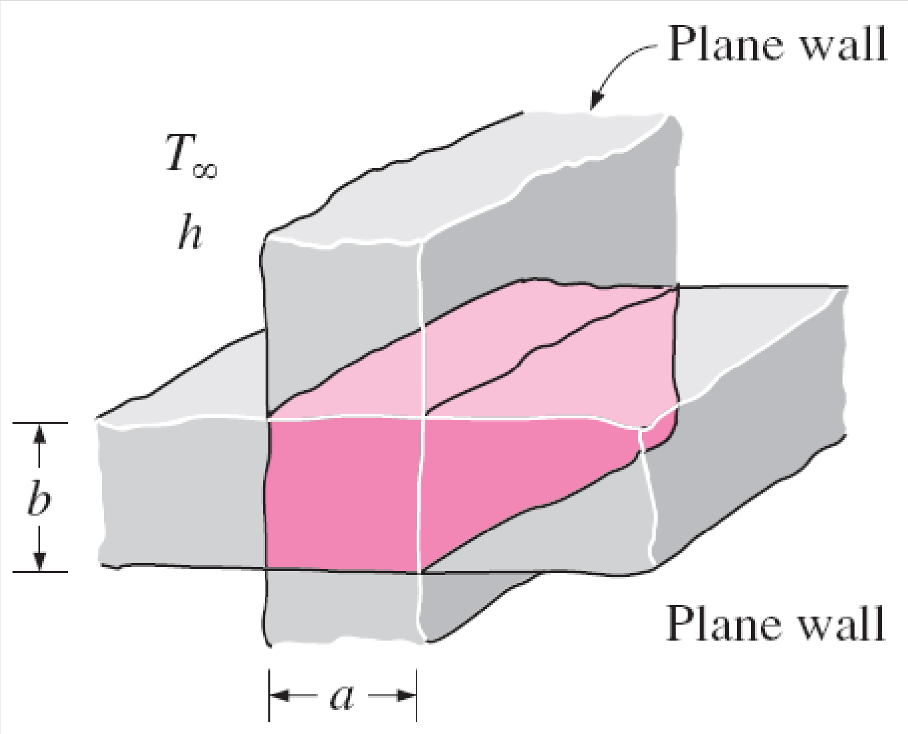
The solution for the two-dimensional short cylinder of height a and radius r_o is equal to the *product* of the nondimensionalized solutions for the one-dimensional plane wall of thickness a and the long cylinder of radius r_o.

$$\left(\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{short}\\ \text{cylinder}}} = \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{plane}\\ \text{wall}}} \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{infinite}\\ \text{cylinder}}}$$



A short cylinder of radius r_o and height a is the *intersection* of a long cylinder of radius r_o and a plane wall of thickness *a.*

$$\left(\frac{T(x,y,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{rectangular}\\ \text{bar}}} = \theta_{\text{wall}}(x,t)\,\theta_{\text{wall}}(y,t)$$



$$\theta_{\text{wall}}(x,t) = \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{plane}\\ \text{wall}}}$$

$$\theta_{\text{cyl}}(r,t) = \left(\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{infinite}\\ \text{cylinder}}}$$

$$\theta_{\text{semi-inf}}(x,t) = \left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{semi-infinite}\\ \text{solid}}}$$

A long solid bar of rectangular profile $a \times b$ is the *intersection* of two plane walls of thicknesses *a* and *b.*

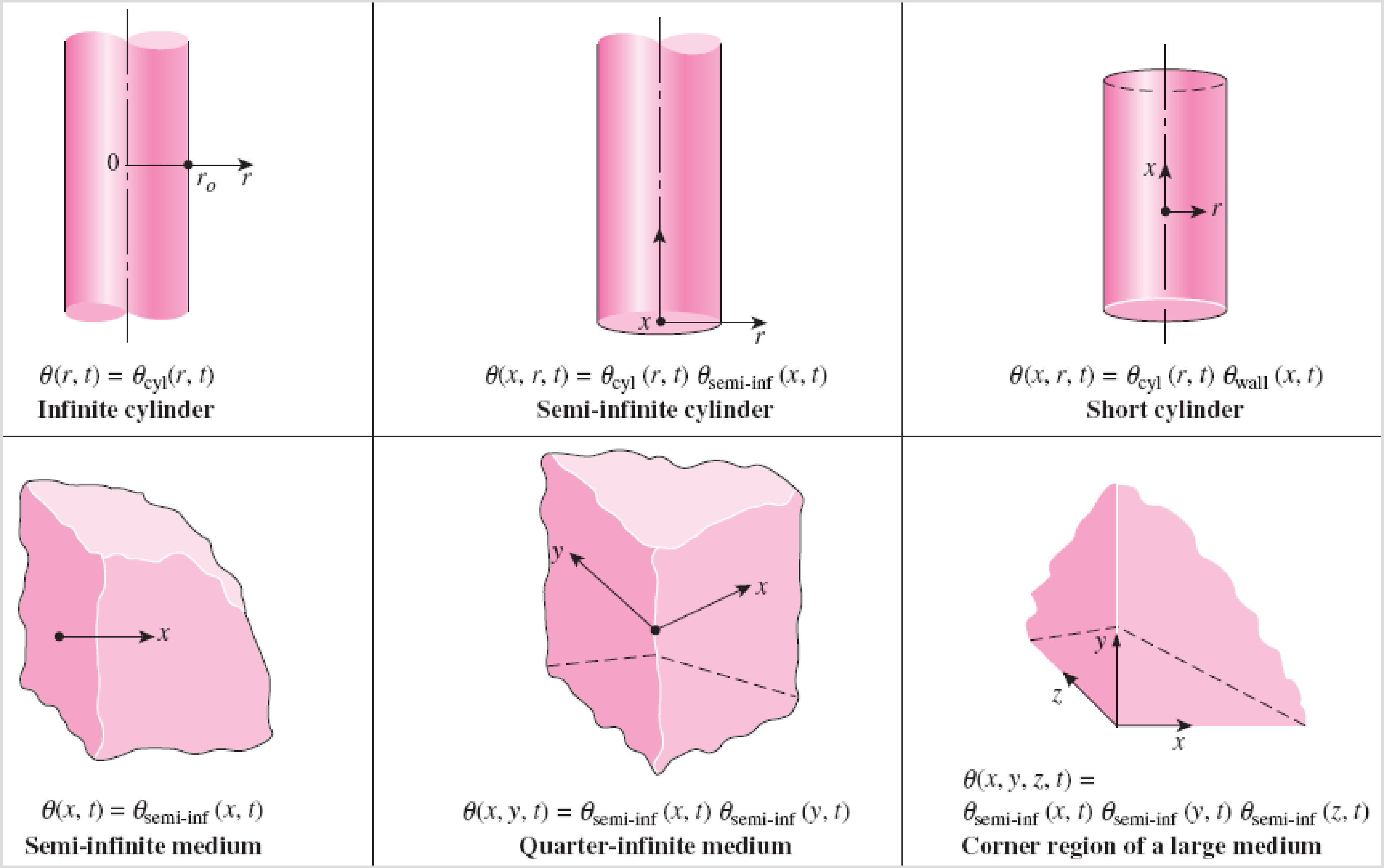
The transient heat transfer for a two-dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total, 2D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]$$

Transient heat transfer for a three-dimensional body formed by the intersection of three one-dimensional bodies 1, 2, and 3 is

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total, 3D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] + \left(\frac{Q}{Q_{\max}}\right)_3 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right]\left[1 - \left(\frac{Q}{Q_{\max}}\right)_2\right]$$

Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞



$\theta(r, t) = \theta_{cyl}(r, t)$
Infinite cylinder

$\theta(x, r, t) = \theta_{cyl}(r, t)\, \theta_{semi\text{-}inf}(x, t)$
Semi-infinite cylinder

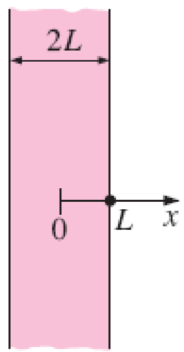
$\theta(x, r, t) = \theta_{cyl}(r, t)\, \theta_{wall}(x, t)$
Short cylinder

$\theta(x, t) = \theta_{semi\text{-}inf}(x, t)$
Semi-infinite medium

$\theta(x, y, t) = \theta_{semi\text{-}inf}(x, t)\, \theta_{semi\text{-}inf}(y, t)$
Quarter-infinite medium

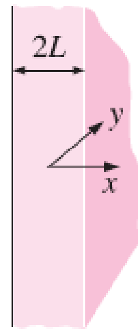
$\theta(x, y, z, t) =$
$\theta_{semi\text{-}inf}(x, t)\, \theta_{semi\text{-}inf}(y, t)\, \theta_{semi\text{-}inf}(z, t)$
Corner region of a large medium

Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature T_i and exposed to convection from all surfaces to a medium at T_∞



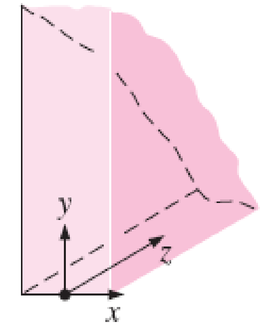
$$\theta(x,t) = \theta_{wall}(x,t)$$

Infinite plate (or plane wall)



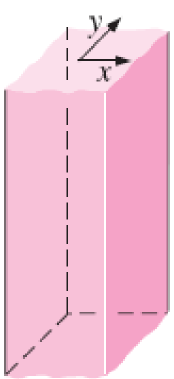
$$\theta(x,y,t) = \theta_{wall}(x,t)\,\theta_{semi\text{-}inf}(y,t)$$

Semi-infinite plate



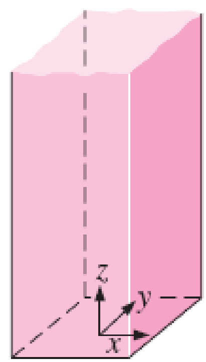
$$\theta(x,y,z,t) = \theta_{wall}(x,t)\,\theta_{semi\text{-}inf}(y,t)\,\theta_{semi\text{-}inf}(z,t)$$

Quarter-infinite plate



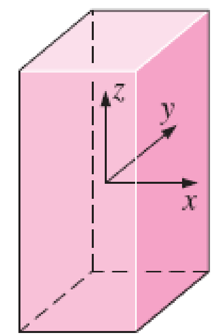
$$\theta(x,y,t) = \theta_{wall}(x,t)\,\theta_{wall}(y,t)$$

Infinite rectangular bar



$$\theta(x,y,z,t) = \theta_{wall}(x,t)\,\theta_{wall}(y,t)\,\theta_{semi\text{-}inf}(z,t)$$

Semi-infinite rectangular bar



$$\theta(x,y,z,t) = \theta_{wall}(x,t)\,\theta_{wall}(y,t)\,\theta_{wall}(z,t)$$

Rectangular parallelepiped

EXAMPLE 4–7 Cooling of a Short Brass Cylinder

A short brass cylinder of diameter $D = 10$ cm and height $H = 12$ cm is initially at a uniform temperature $T_i = 120°C$. The cylinder is now placed in atmospheric air at 25°C, where heat transfer takes place by convection, with a heat transfer coefficient of $h = 60$ W/m$^2 \cdot$ °C. Calculate the temperature at (*a*) the center of the cylinder and (*b*) the center of the top surface of the cylinder 15 min after the start of the cooling.

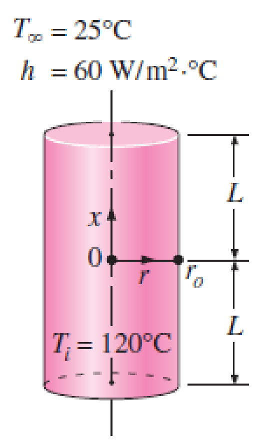


FIGURE 4–28
Schematic for Example 4–7.

SOLUTION A short cylinder is allowed to cool in atmospheric air. The temperatures at the centers of the cylinder and the top surface are to be determined.

Assumptions **1** Heat conduction in the short cylinder is two-dimensional, and thus the temperature varies in both the axial *x*- and the radial *r*-directions. **2** The thermal properties of the cylinder and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of brass at room temperature are $k = 110$ W/m · °C and $\alpha = 33.9 \times 10^{-6}$ m²/s (Table A-3). More accurate results can be obtained by using properties at average temperature.

Analysis (*a*) This short cylinder can physically be formed by the intersection of a long cylinder of radius $r_o = 5$ cm and a plane wall of thickness $2L = 12$ cm, as shown in Fig. 4–28. The dimensionless temperature at the center of the plane wall is determined from Figure 4–13*a* to be

$$\left.\begin{aligned} \tau &= \frac{\alpha t}{L^2} = \frac{(3.39 \times 10^{-5}\ \text{m}^2/\text{s})(900\ \text{s})}{(0.06\ \text{m})^2} = 8.48 \\ \frac{1}{\text{Bi}} &= \frac{k}{hL} = \frac{110\ \text{W/m} \cdot {}^\circ\text{C}}{(60\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.06\ \text{m})} = 30.6 \end{aligned}\right\} \theta_{\text{wall}}(0, t) = \frac{T(0, t) - T_\infty}{T_i - T_\infty} = 0.8$$

Similarly, at the center of the cylinder, we have

$$\left.\begin{aligned} \tau &= \frac{\alpha t}{r_o^2} = \frac{(3.39 \times 10^{-5}\ \text{m}^2/\text{s})(900\ \text{s})}{(0.05\ \text{m})^2} = 12.2 \\ \frac{1}{\text{Bi}} &= \frac{k}{hr_o} = \frac{110\ \text{W/m}\cdot{}^\circ\text{C}}{(60\ \text{W/m}^2\cdot{}^\circ\text{C})(0.05\ \text{m})} = 36.7 \end{aligned}\right\} \theta_{\text{cyl}}(0, t) = \frac{T(0, t) - T_\infty}{T_i - T_\infty} = 0.5$$

Therefore,

$$\left(\frac{T(0, 0, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{short}\\ \text{cylinder}}} = \theta_{\text{wall}}(0, t) \times \theta_{\text{cyl}}(0, t) = 0.8 \times 0.5 = 0.4$$

and

$$T(0, 0, t) = T_\infty + 0.4(T_i - T_\infty) = 25 + 0.4(120 - 25) = 63^\circ\text{C}$$

This is the temperature at the center of the short cylinder, which is also the center of both the long cylinder and the plate.

(*b*) The center of the top surface of the cylinder is still at the center of the long cylinder ($r = 0$), but at the outer surface of the plane wall ($x = L$). Therefore, we first need to find the surface temperature of the wall. Noting that $x = L = 0.06$ m,

$$\left.\begin{aligned} \frac{x}{L} &= \frac{0.06\ \text{m}}{0.06\ \text{m}} = 1 \\ \frac{1}{\text{Bi}} &= \frac{k}{hL} = \frac{110\ \text{W/m}\cdot{}^\circ\text{C}}{(60\ \text{W/m}^2\cdot{}^\circ\text{C})(0.06\ \text{m})} = 30.6 \end{aligned}\right\} \frac{T(L, t) - T_\infty}{T_o - T_\infty} = 0.98$$

Then

$$\theta_{\text{wall}}(L, t) = \frac{T(L, t) - T_\infty}{T_i - T_\infty} = \left(\frac{T(L, t) - T_\infty}{T_o - T_\infty}\right)\left(\frac{T_o - T_\infty}{T_i - T_\infty}\right) = 0.98 \times 0.8 = 0.784$$

Therefore,

$$\left(\frac{T(L, 0, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{short}\\ \text{cylinder}}} = \theta_{\text{wall}}(L, t)\theta_{\text{cyl}}(0, t) = 0.784 \times 0.5 = 0.392$$

and

$$T(L, 0, t) = T_\infty + 0.392(T_i - T_\infty) = 25 + 0.392(120 - 25) = 62.2^\circ\text{C}$$

which is the temperature at the center of the top surface of the cylinder.

EXAMPLE 4–8 Heat Transfer from a Short Cylinder

Determine the total heat transfer from the short brass cylinder (ρ = 8530 kg/m^3, C_p = 0.380 kJ/kg · °C) discussed in Example 4–7.

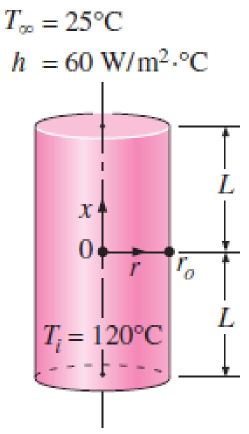


FIGURE 4–28
Schematic for Example 4–7.

SOLUTION We first determine the maximum heat that can be transferred from the cylinder, which is the sensible energy content of the cylinder relative to its environment:

$$m = \rho V = \rho \pi r_o^2 L = (8530 \text{ kg/m}^3)\pi(0.05 \text{ m})^2(0.06 \text{ m}) = 4.02 \text{ kg}$$

$$Q_{\max} = mC_p(T_i - T_\infty) = (4.02 \text{ kg})(0.380 \text{ kJ/kg}\cdot{}^\circ\text{C})(120 - 25)^\circ\text{C} = 145.1 \text{ kJ}$$

Then we determine the dimensionless heat transfer ratios for both geometries. For the plane wall, it is determined from Fig. 4–13*c* to be

$$\left.\begin{aligned} \text{Bi} &= \frac{1}{1/\text{Bi}} = \frac{1}{30.6} = 0.0327 \\ \frac{h^2\alpha t}{k^2} &= \text{Bi}^2\tau = (0.0327)^2(8.48) = 0.0091 \end{aligned}\right\} \left(\frac{Q}{Q_{\max}}\right)_{\substack{\text{plane}\\\text{wall}}} = 0.23$$

Similarly, for the cylinder, we have

$$\left.\begin{aligned} \text{Bi} &= \frac{1}{1/\text{Bi}} = \frac{1}{36.7} = 0.0272 \\ \frac{h^2\alpha t}{k^2} &= \text{Bi}^2\tau = (0.0272)^2(12.2) = 0.0090 \end{aligned}\right\} \left(\frac{Q}{Q_{\max}}\right)_{\substack{\text{infinite}\\\text{cylinder}}} = 0.47$$

Then the heat transfer ratio for the short cylinder is, from Eq. 4–28,

$$\begin{aligned} \left(\frac{Q}{Q_{\max}}\right)_{\text{short cyl}} &= \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2 \left[1 - \left(\frac{Q}{Q_{\max}}\right)_1\right] \\ &= 0.23 + 0.47(1 - 0.23) = 0.592 \end{aligned}$$

Therefore, the total heat transfer from the cylinder during the first 15 min of cooling is

$$Q = 0.592 Q_{\max} = 0.592 \times (145.1 \text{ kJ}) = \mathbf{85.9 \text{ kJ}}$$

EXAMPLE 4–9 Cooling of a Long Cylinder by Water

A semi-infinite aluminum cylinder of diameter $D = 20$ cm is initially at a uniform temperature $T_i = 200°C$. The cylinder is now placed in water at 15°C where heat transfer takes place by convection, with a heat transfer coefficient of $h = 120 \text{ W/m}^2 \cdot °C$. Determine the temperature at the center of the cylinder 15 cm from the end surface 5 min after the start of the cooling.

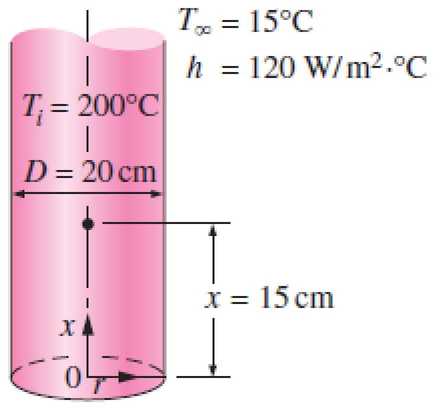


FIGURE 4–29
Schematic for Example 4–9.

SOLUTION A semi-infinite aluminum cylinder is cooled by water. The temperature at the center of the cylinder 15 cm from the end surface is to be determined.

Assumptions **1** Heat conduction in the semi-infinite cylinder is two-dimensional, and thus the temperature varies in both the axial x- and the radial r-directions. **2** The thermal properties of the cylinder and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of aluminum at room temperature are $k = 237$ W/m · °C and $\alpha = 9.71 \times 10^{-6}$ m^2/s (Table A-3). More accurate results can be obtained by using properties at average temperature.

Analysis This semi-infinite cylinder can physically be formed by the intersection of an infinite cylinder of radius $r_0 = 10$ cm and a semi-infinite medium, as shown in Fig. 4–29.

We will solve this problem using the one-term solution relation for the cylinder and the analytic solution for the semi-infinite medium. First we consider the infinitely long cylinder and evaluate the Biot number:

$$\text{Bi} = \frac{hr_o}{k} = \frac{(120\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.1\ \text{m})}{237\ \text{W/m} \cdot {}^\circ\text{C}} = 0.05$$

The coefficients λ_1 and A_1 for a cylinder corresponding to this Bi are determined from Table 4–1 to be $\lambda_1 = 0.3126$ and $A_1 = 1.0124$. The Fourier number in this case is

$$\tau = \frac{\alpha t}{r_o^2} = \frac{(9.71 \times 10^{-5}\ \text{m}^2/\text{s})(5 \times 60\ \text{s})}{(0.1\ \text{m})^2} = 2.91 > 0.2$$

and thus the one-term approximation is applicable. Substituting these values into Eq. 4–14 gives

$$\theta_0 = \theta_{\text{cyl}}(0, t) = A_1 e^{-\lambda_1^2 \tau} = 1.0124 e^{-(0.3126)^2(2.91)} = 0.762$$

The solution for the semi-infinite solid can be determined from

$$1 - \theta_{\text{semi-inf}}(x, t) = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\left[\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

First we determine the various quantities in parentheses:

$$\xi = \frac{x}{2\sqrt{\alpha t}} = \frac{0.15\ \text{m}}{2\sqrt{(9.71 \times 10^{-5}\ \text{m}^2/\text{s})(5 \times 60\ \text{s})}} = 0.44$$

$$\frac{h\sqrt{\alpha t}}{k} = \frac{(120\ \text{W/m}^2 \cdot {}^\circ\text{C})\sqrt{(9.71 \times 10^{-5}\ \text{m}^2/\text{s})(300\ \text{s})}}{237\ \text{W/m} \cdot {}^\circ\text{C}} = 0.086$$

$$\frac{hx}{k} = \frac{(120\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.15\ \text{m})}{237\ \text{W/m} \cdot {}^\circ\text{C}} = 0.0759$$

$$\frac{h^2 \alpha t}{k^2} = \left(\frac{h\sqrt{\alpha t}}{k}\right)^2 = (0.086)^2 = 0.0074$$

Substituting and evaluating the complementary error functions from Table 4–3,

$$\begin{aligned}\theta_{\text{semi-inf}}(x, t) &= 1 - \text{erfc}\,(0.44) + \exp\,(0.0759 + 0.0074)\ \text{erfc}\,(0.44 + 0.086)\\ &= 1 - 0.5338 + \exp\,(0.0833) \times 0.457\\ &= 0.963\end{aligned}$$

Now we apply the product solution to get

$$\left(\frac{T(x, 0, t) - T_\infty}{T_i - T_\infty}\right)_{\substack{\text{semi-infinite}\\ \text{cylinder}}} = \theta_{\text{semi-inf}}(x, t)\,\theta_{\text{cyl}}(0, t) = 0.963 \times 0.762 = 0.734$$

and

$$T(x, 0, t) = T_\infty + 0.734(T_i - T_\infty) = 15 + 0.734(200 - 15) = 151^\circ\text{C}$$

which is the temperature at the center of the cylinder 15 cm from the exposed bottom surface.

EXAMPLE 4–10 Refrigerating Steaks while Avoiding Frostbite

In a meat processing plant, 1-in.-thick steaks initially at 75°F are to be cooled in the racks of a large refrigerator that is maintained at 5°F (Fig. 4–30). The steaks are placed close to each other, so that heat transfer from the 1-in.-thick edges is negligible. The entire steak is to be cooled below 45°F, but its temperature is not to drop below 35°F at any point during refrigeration to avoid "frostbite." The convection heat transfer coefficient and thus the rate of heat transfer from the steak can be controlled by varying the speed of a circulating fan inside. Determine the heat transfer coefficient h that will enable us to meet both temperature constraints while keeping the refrigeration time to a minimum. The steak can be treated as a homogeneous layer having the properties ρ = 74.9 lbm/ft^3, C_p = 0.98 Btu/lbm · °F, k = 0.26 Btu/h · ft · °F, and α = 0.0035 ft^2/h.

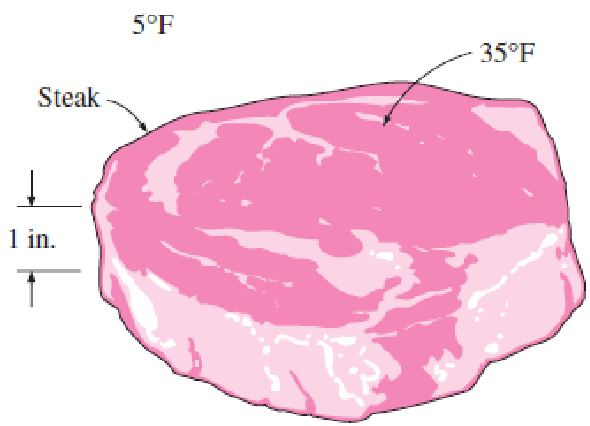


FIGURE 4–30
Schematic for Example 4–10.

SOLUTION Steaks are to be cooled in a refrigerator maintained at 5°F. The heat transfer coefficient that will allow cooling the steaks below 45°F while avoiding frostbite is to be determined.

Assumptions **1** Heat conduction through the steaks is one-dimensional since the steaks form a large layer relative to their thickness and there is thermal symmetry about the center plane. **2** The thermal properties of the steaks and the heat transfer coefficient are constant. **3** The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

Properties The properties of the steaks are as given in the problem statement.

Analysis The lowest temperature in the steak will occur at the surfaces and the highest temperature at the center at a given time, since the inner part will be the last place to be cooled. In the limiting case, the surface temperature at $x = L = 0.5$ in. from the center will be 35°F, while the midplane temperature is 45°F in an environment at 5°F. Then, from Fig. 4–13*b*, we obtain

$$\left.\begin{aligned} \frac{x}{L} &= \frac{0.5\text{ in.}}{0.5\text{ in.}} = 1 \\ \frac{T(L,t) - T_\infty}{T_o - T_\infty} &= \frac{35-5}{45-5} = 0.75 \end{aligned}\right\} \quad \frac{1}{\text{Bi}} = \frac{k}{hL} = 1.5$$

which gives

$$h = \frac{1}{1.5}\frac{k}{L} = \frac{0.26\text{ Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}}{1.5(0.5/12\text{ ft})} = 4.16\text{ Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F}$$

Discussion The convection heat transfer coefficient should be kept below this value to satisfy the constraints on the temperature of the steak during refrigeration. We can also meet the constraints by using a lower heat transfer coefficient, but doing so would extend the refrigeration time unnecessarily.